SOME ASPECTS OF REPRESENTING UNCERTAINTY IN NONLINEAR SYSTEMS

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ABSTRACT:
Informations of any kind are characterized on one hand by the amount of its uncertainty, inaccuracy. The method of representation of this uncertainty must be not only uniform, but in harmony with the modeling and information processing method. With the appearance and spreading of new modeling and computing techniques, the computing and representation of uncertainty in different systems becomes an increasingly important question. This paper deals with the limits of the „classical", probability theory based uncertainty representation, and examines some possible new, fuzzy based uncertainty representation methods.

Keywords: uncertainty representation, fuzzy theory, probability theory, nonlinear systems.

1. INTRODUCTION

The information coming from our environment - including the results of measurement procedures - usually are not perfectly reliable, they involve a certain amount of uncertainty, inaccuracy. The degree of this uncertainty is a very important additional information, because we can decide only in possession of this knowledge, how valid the information is, in what degree and for what purpose is it usable. Unfortunately, this uncertainty and inaccuracy of the data is often overlooked in case of „non engineering", or rather „non measuring" approaches. The problem becomes even heavier because of the fact, that the information often comes through complex, nonlinear channels, or further transformations may be necessary during the data processing, and some results are computed based on different, possible uncertain input data. So it is important to have a uniform method for the representation and handling of uncertainty, by aid whereof the resultant uncertainty can be computed during the information process.

This uncertainty, inaccuracy of the information comes partly from the incompleteness or noisiness of the available data, and partly from the inaccuracy of the measuring and information processing system. The model used to describe the system limits the achievable accuracy of the obtainable data. This limitation comes both from the accuracy and validity of the model, and the data representation method, which have a great influence on the information processing methods. This latter factor can particularly influence, by what means and how exactly the uncertainty of data is expressible. Nowadays a characteristic tendency is the appearance and spreading of new modeling and computing techniques. These techniques often use different data representations and with the appearance of complex, „hybrid" systems the communication between systems, based on different modeling techniques becomes an increasingly important question. Thus the question arises, that in the case of different data representations how can we express and compute the uncertainty of the data, and how comparable are these different measures of uncertainty with each other (see [1]).

When we choose an uncertainty representation method, we have to consider more aspects of the problem. Essential expectation is, that the measure of uncertainty has to be analogous with the subjective opinion, i. e. the measure of uncertainty of subjectively „more uncertain", „less accurate" data has a higher value. The uniformity, popularity is also important, because the uncertainty of the information must be interpretable, appraisable by anybody.

Important aspects are the nature and characteristics of the information processing system, as well. In the case of a simple, „classical" computing, information processing methods the well-proved and theoretically well established probability theory is usable, but in the case of complex, non-linear or „soft" computing methods it can be more suitable - or even necessary - to find other representation forms. These representations should cope with the data representation, resulting in a more accurate uncertainty expression and more comfortable usage. On the other hand, sometimes the type of information and uncertainty makes rather difficult or even impossible the probability theory-based representation (e. g. uncertainty originated from incomplete data or subjective information sources; the accuracy of an estimated output variable).

However, we can not forget, that the different representation methods have to be comparable, convertible. Thus the new representation methods have to be comparable with the „classical" probability theory-based methods.

The aim of this paper is to establish the examination of this topic. Section II. deals with the „classical", probability theory based representation and with the connected standards. In Section III. the limits of this representation method are investigated. Section IV. describes some new, fuzzy based representation methods, while Section V. analyzes the comparability, convertibility of these new methods and the „classical" techniques.

2. UNCERTAINTY REPRESENTATION BASED ON PROBABILITY THEORY

In the practice the probability theory based uncertainty representation is wide-spread used for the expression of the uncertainty of the information. The advantages of this method are the uniformity, popularity and the profound theoretical background [2]. The standards and recommendations regarding the expression of measurement uncertainty are also linked with this method.

Traditionally two main types of measurement error were considered: the so-called random and systematic errors. The effects of random errors can be reduced by repeated measurement and statistical methods, while the systematic ones by deterministic corrections. Simultaneously in the recently wide-spread standards ([3]) a different approach has appeared: the so-called 'A' and 'B' type uncertainty. In both cases the uncertainty is expressed as variance or standard deviation. For category 'A' the evaluation based only on the statistical evaluation of the measurement, while for category 'B' also on a priory information, like previous measured data, specifications of the instruments, calibration data, etc.

In the case of the probability theory based uncertainty representation there are known methods for the expression of the resultant uncertainty in nearly linear systems. These methods are based on the first-order Taylor expansion of the system. If f maps the input variables x_1, x_2, ..., x_N - with known standard deviation - into the output variable y, the resultant uncertainty can be assessed according to the followings:

$$d_y^2 \cong \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 d_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} r_{i,j}\,, \tag{1}$$

where d_i stands for the standard deviation of x_i , and $r_{i,j}$ denotes the covariance between x_i and x_j .

Sometimes it is necessary to give the measurement uncertainty with an interval, which includes a great part of the - hypothetical - distribution of the measured data (see [4]). The width of this interval (in the standards: extended measurement uncertainty) is computed from the standard deviation (in the standards: standard measurement uncertainty) with the help of a so-called extension factor:

$$U_x = k * d_x, \tag{2}$$

where U_x stands for the half-width of the interval, and d_x denotes the standard deviation. If all components of the uncertainty is 'A' type, then this interval is equivalent with a confidence interval with a given confidence level. The extension factor 'k' depends on the confidence level and the type of the distribution. In the practice - with a supposed normal distribution - often used values are k=2 (confidence level: 95%) and k=3 (confidence level: 99%) .

In case of non-linear transformations the computing of the confidence intervals is usually not too reliable, because in such cases the nearly normal distribution of data is not guaranteed, and the actual distribution is not or only with difficulties obtainable, so the extension factor is not or only approximately computable. On the other hand, to compute the confidence intervals we use the resultant standard deviation, thus it is an important question, how accurate the computing of the resultant standard deviation can be in non-linear systems.

3. THE LIMITS OF THE PROBABILITY THEORY BASED UNCERTAINTY REPRESENTATION

For the examination of the probability theory based uncertainty representation the MATLAB package [5] was used. By simulating different non-linear characteristics the computed standard deviation of the output variable (from the 1. formula) and the variance got from the simulation were compared.

It was proved, that while in nearly linear cases, where the rise of the characteristic changes only slowly and doesn't change sign, the above mentioned formulas and the simulations give nearly the same values, but in strongly nonlinear cases the formulas do not give accurate results, in fact sometimes they are not at all suitable to compute the resultant uncertainty.

Figure 1. shows an example for the case of „abrupt" characteristics, where not only the function describing the system, but also its derivative changes fast, there is a great deviation between the variance got from the simulation and the resultant standard deviation computed from the formulas. Sometimes even the statistical average of the transformed variables considerably deviates from the value got by the transformation of the expected value.

In case of non-monotone characteristics (e. g. sine function, see Figure 2.) the computed value and the values got from the simulation greatly differ, as well. This deviation depends not only on the standard deviation of the input variable(s), but also on its (their) expected value(s). So the theoretical formulas give inaccurate result in the case of some given output expected value, even if the standard variable is small (Figure 2.).

One solution for these problems can be the improving of the formulas - maybe by taking into consideration the higher-order derivatives, as well. But this would make the computing more complex, and not necessary result in better results e. g. in the case of systems, where also the higher-order derivatives change fast or are non-monotone. So it is worth considering, to use other modeling techniques for the representation of the uncertainty in non-linear systems.

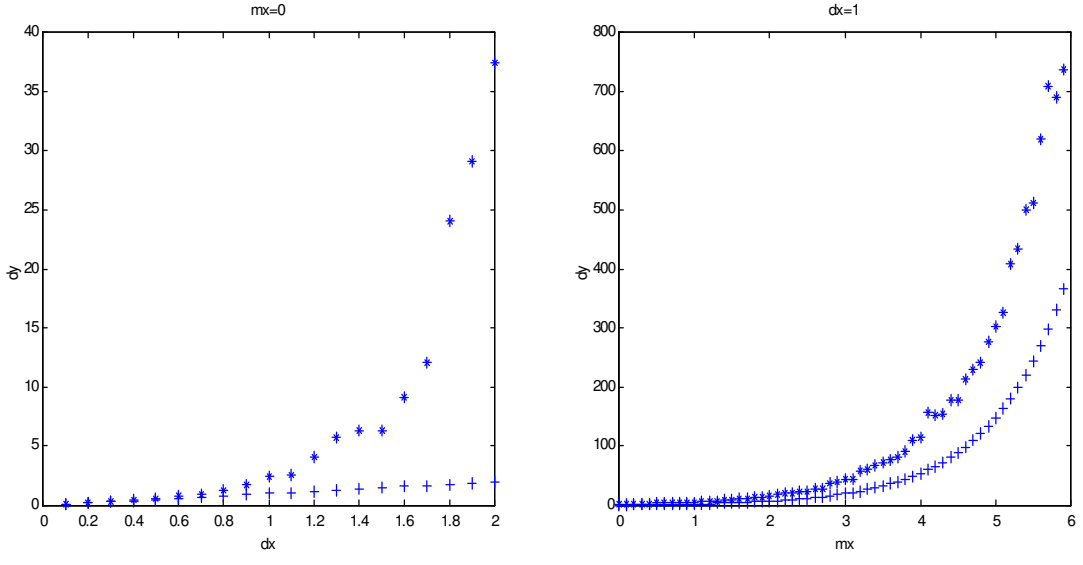

Figure 1.: The effects of exponential transformation on the variance and the standard deviation
++++++ : values computed from the (1) formula
********: values got from the simulation

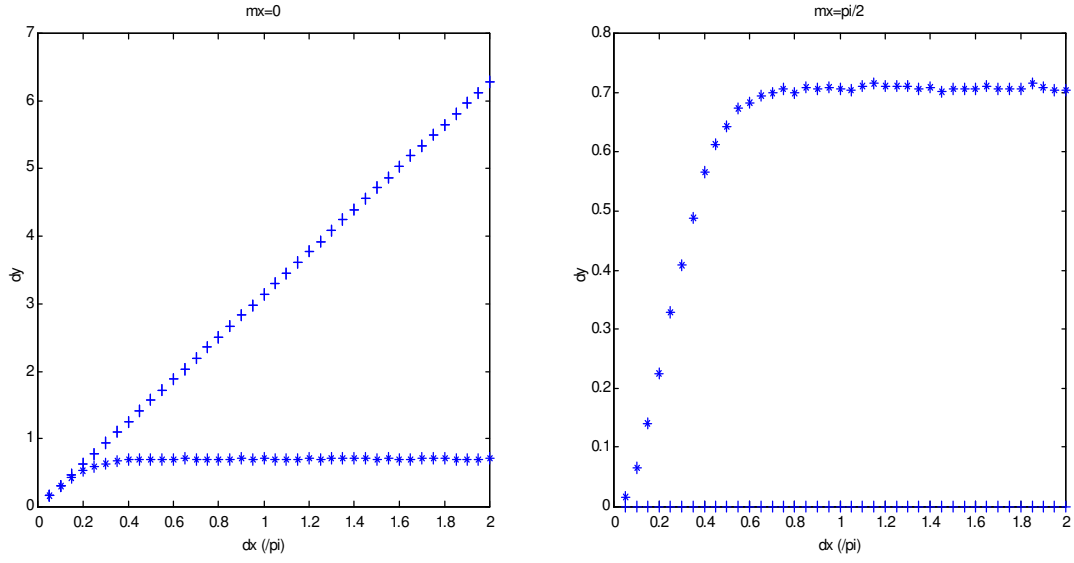

Figure 2.: The effects of sinusoid transformation on the variance and the standard deviation
++++++ : values computed from the (1) formula
********: values got from the simulation

An other, not negligible aspect is, that today's new - often embedded or real-time - systems raise new problems, which can only partly, or not at all treated by the use of the „classical", probability theory based uncertainty representation and handling methods. Besides the difficulties in modeling, and complexity, such problems can be the handling and measure of incomplete information, or the maintenance of information processing in case of lost data. These questions call for the application of new tools, among which the so-called „soft" - fuzzy, neural, genetic -computing systems are increasingly wide-spread used. These methods are often based on modeling methods, which are completely different from the probability theory, so the „classical" tools for uncertainty representation are not usable. Furthermore, a transformation implemented by a complicated, complex system is often not describable by a simple function, so the derivative can be not or only approximately computed. In these systems the representation of the uncertainty during the data processing has not been yet solved, so the authenticity, accuracy of the resultant data is not known.

4. UNCERTAINTY REPRESENTATION BASED ON FUZZY THEORY

The most promising new modeling method seems the fuzzy modeling. Fuzzy theory is often used to solve modeling and computing tasks, so with the fuzzy based uncertainty representation we can solve two problems. First, there is a possibility, that we can improve the accuracy of the computing of the resultant uncertainty in non-linear systems. Secondly, we can easily represent and compute the uncertainty in fuzzy based systems, where the probability theory can be used only with difficulties. Another great advantage of the fuzzy representation is, that it is close to human thinking, and the effect of parameter changes can be easily estimated, so the fuzzy based systems are usually easy to survey.

Furthermore, some kind of uncertainty (e. g. incomplete data or subjective information) can be expressed far better with fuzzy sets, then with probabilistic or statistical methods.

In case of fuzzy representation of the information, data is expressed not by a concrete value („measured data") and the degree of the connected uncertainty (variance, standard deviation, confidence-interval), nor with an -approximate - probability distribution, but with a membership function, which gives, that the elements of the universal set in what degree belong to the set, which contains the value(s) of the measured data. This method can be used to express non statistical type - e. g. originated from subjective data - uncertainty, as well.

To express the data we can use fuzzy numbers, i. e. fuzzy sets defined on $\Re$, which are normalized (at least one of its elements attains the „1" membership grade) and convex:

$$\mu(\lambda r+(1-\lambda)s) \geq \min(\mu(r),\mu(s)) \qquad \forall r,s \in R;\ \lambda \in [0,1] \tag{3}$$

If during the information process we use fuzzy data representation, then, if we success to define a degree of uncertainty to the membership function, the uncertainty would become expressible and measurable at any point of the system. The fuzzy data representation is solved in some „soft" computing systems, such as fuzzy, fuzzy-neural systems. In case of "classical" data-processing methods the membership function of the output variable y is computable from the membership functions of the input variables x_1, x_2, ..., x_N with the help of the extension principle:

$$\mu(y) = \sup\{\mu_1(x_1) * \mu_2(x_2) * ... * \mu_N(x_N) \,|\, y = f(x_1, x_2, ..., x_N)\}. \tag{4}$$

If we want to change to the „classical" data representation (e. g. in the end of the data processing we need a concrete value), we have to defuzzyficate the fuzzy number. Different defuzzyfication methods are known, here the center of gravity (CoG) method will be used, because of the analogy with the probability theory, and because this method possibly may express most faithfully the information involved in the membership function. With the center of gravity method we can compute the defuzzyficated value accordingly:

$$x^* = \frac{\int x\mu(x)dx}{\int \mu(x)dx}. \tag{5}$$

Although in the literature (see e. g. [6]) there are some measures for the measurement of the uncertainty of fuzzy sets (e. g. U-uncertainty), but these measures are based on discrete domain, where no distance is defined or considered between the discrete values. Thus, though these measures can be very useful in the expression of the uncertainty in case of a diagnostical tasks, they are not suitable for the expression of measurement uncertainty.

In the following three methods are examined, avoiding this problem:

I. Width of α-cut

The uncertainty of a fuzzy number can be expressed by the width of the interval, where the membership function is higher or equal then a given α value. Since the membership function is convex, this interval can't be „holed" (every α-cut is convex, too).

$$u_{1,\alpha} = x_2 - x_1, \text{ where } x_1 = \inf\{x \,|\, \mu(x) \geq \alpha\} \text{ and } x_2 = \sup\{x \,|\, \mu(x) \geq \alpha\}. \tag{6}$$

II. Integral of the membership function

The area under the curve can also express the uncertainty, since wider, higher membership function result a higher value. Theoretically there can be cases, when data with subjectively different uncertainty get the same value, but if we use only fuzzy numbers, these cases can be excluded.

$$u_2 = \int_{-\infty}^{+\infty} \mu(x)dx \tag{7}$$

III. Quadratic deviation from the defuzzyficated value

This method is well-known from the probability theory, the value got in this way is similar to the standard deviation.

$$u_3^2 = \int_{-\infty}^{+\infty} (x^* - x)^2 \mu(x)dx, \tag{8}$$

where x^* stands for the defuzzyficated value.

5. EXAMINATION OF FUZZY BASED UNCERTAINTY REPRESENTATION METHODS

The measure of uncertainty must be not only analogous with the subjective opinion (e. g. data represented by wider membership function must have a higher uncertainty value), but also must be comparable with the probability theory based representation, and it must behave similarly to the „classical" measures during the different transformations. It is necessary, partly because the probability theory based representation is wide-spread, and it is favorable for the users, if the new measure of uncertainty is comparable, convertible with the „classical" techniques. On the other hand, we must ensure the communication between systems based on different modeling techniques, so it is important to have some „rule of conversion" between the different measures of uncertainty.

In this paper three aspects of this question of comparability, convertibility are examined:

1. The effects of linear transformations to the measure of uncertainty
2. Relations between the fuzzy measures of uncertainty and the standard deviation
3. Relations between the fuzzy measures of uncertainty and the standard deviation after different transformations

In the practice the normal distribution is often used, as supposed distribution of the variables, so for the comparisons random variables with normal distribution and the corresponding fuzzy numbers are used.

5.1. The effects of linear transformations to the measure of uncertainty

It is known, that in case of a linear transformation, given by the

$$y = ax + b \tag{9}$$

formula, it is true for the standard deviation of the variables x and y, that

$$d_y = |a| d_x . \tag{10}$$

This is the same, as the subjective expectation, thus it is necessary, that the new measures of uncertainty also fulfill this formula. Since this transformation -if $a \neq 0$ - is a one-to-one mapping, we got from the extension principle (equation (4)):

$$\mu_y(y) = \mu_x\left(\frac{y-b}{a}\right) \tag{11}$$

I. Width of α-cut

From the (11) formula:

$$\mu_x(x_1) \geq \alpha \Leftrightarrow \mu_y(ax_1 + b) \geq \alpha . \tag{12}$$

Thus, if the α-cut of the input fuzzy number is $[x_1, x_2]$, with $x_2 - x_1$ width, then the α-cut of the output variable will be $[ax_1 + b, ax_2 + b]$ - if a>0 - or $[ax_2 + b, ax_1 + b]$ - if a<0 - with $|a|(x_2 - x_1)$ width.

II. Integral of the membership function

With the $y = ax + b$ substitution we get:

$$u_2(y) = \int_{-\infty}^{+\infty} \mu_y(y)dy = |a| \int_{-\infty}^{+\infty} \mu_x(x)dx = |a| u_2(x) \tag{13}$$

III. Quadratic deviation from the defuzzyficated value

With the $y = ax + b$ substitution we get:

$$y^* = \frac{\int y\mu_y(y)dy}{\int \mu_y(y)dy} = \frac{a^2 \int x\mu_x(x)dx + ab\int \mu_x(x)dx}{a\int \mu_x(x)dx} = ax^* + b \tag{14}$$

$$u_3^2(y) = \int (y^* - y)^2 \mu_y(y)dy = |a|^3 \int (x^* - x)^2 \mu_x(x)dx = |a|^3 u_3^2(x) \tag{15}$$

As it appeared, in the case of u_1 and u_2 the measures of uncertainty behave in the same way, as the „classical" standard deviation. But it is not true any more about the method of quadratic deviation.

5.2. Relations between the fuzzy measures of uncertainty and the standard deviation

The membership function of a fuzzy number, corresponding to a normal random variable with an expected value of $\bar{x}$ and standard deviation of σ, can be computed over a disrete universe as [7] (see Figure 3.):

$$\mu(\frac{x_i + x_{i-1}}{2}) = \sum_{k=1}^{N} \min\{\lambda_i, \lambda_k\}, \text{ where} \qquad (16)$$

$$\lambda_i = \int_{x_{i-1}}^{x_i} \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\bar{x})^2}{2\sigma^2})dx,$$

and $x_0 = -\infty$, $x_N = +\infty$, $x_i = \bar{x} - 3\sigma + \frac{(i-1)6\sigma}{N-2}$.

With the help of these formulas we can examine the relation between the standard deviation and the uncertainty of the corresponding fuzzy number (Figure 4.). It is preferred, if there is a linear relationship between the uncertainty measures, they could be converted easily to each other. This expectation is fulfilled in the case of u_1 and u_2, but not in the case of u_3.

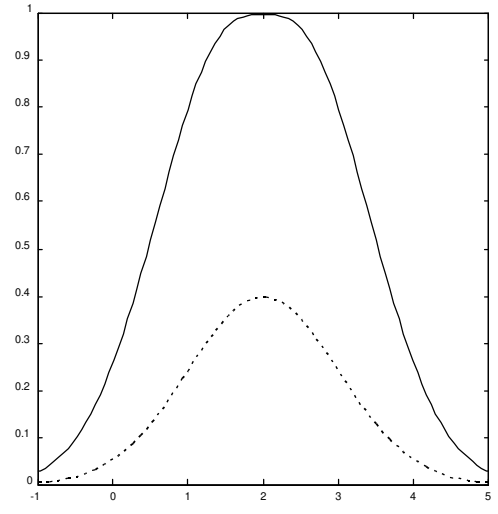

Figure 3.: A normal distribution function and the membership function of the corresponding fuzzy number

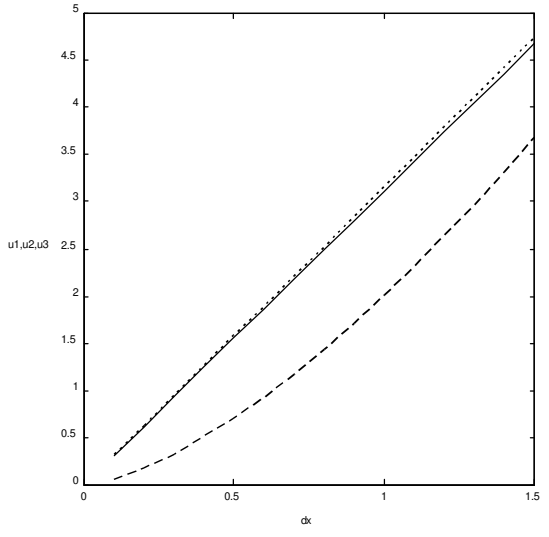

Figure 4.: ----: u_1 (α -cut, α=0,5)
.....: u_2 (integral)
- - -: u_3 (quadratic deviation)

5.3. Relation between the fuzzy measures of uncertainty and the standard deviation after different transformations

One other expectation can be, that there must be a simple relation between the standard deviation and the uncertainty of the corresponding fuzzy number in case of different transformations, as well. It is preferred, if this relation is linear, with the same - or similar - factor. This factor must be the same - or similar - to the factor computed from the input variables.

The results are summarized in tables 1.-2.: 'a' is the „optimal" factor of the linear relationship (where $\sum(u_i - ad)^2$ is minimal), e is the relative quadratic error (e=$\sum(\frac{u_i - ad}{ad})^2$), and e_0 is the relative

quadratic error if we estimate the output fuzzy uncertainty with $a_o d$ where a_0 is the „optimal" factor for the input variables (u_1: a_0=3.1224; u_2: a_0=3.1606; u_3: a_0=2.1357). It is preferred, that the 'a' factors are nearly the same in the case of different transformations, and e and e_0 are small.

As it appears from these data, in the case of u_1 and u_2, there can exist a linear conversion between the standard deviation and the fuzzy uncertainty ($u_i = ad$) with a relatively small error. In the case of u_3 there are no linear relationship and the u_3/d rate changes greatly during the transformations.

	x m_x=1	x+y m_x=1,m_y=2 d_x=0.1	x+y m_x=1,m_y=2 d_x=d_y	x*y m_x=2,m_y=1 d_x=0.1	x*y m_x=5,m_y=1 d_x=0.1	x*y m_x=2,m_y=1 d_x=d_y	1/x m_x=10	x^2 m_x=1	x^2 m_x=5
u_1 a	3.1224	3.1814	3.3442	3.321	3.3289	3.5611	3.0774	2.6592	3.0948
e	0	0.009	0	0.0117	0.0104	0.0566	0.0028	0.1337	0.0012
e_0	0	0.021	0.0757	0.0588	0.0617	0.2025	0.0035	0.1964	0.0016
u_2 a	3.1606	3.2002	3.4286	3.2987	3.3743	3.7129	3.1677	2.8861	3.1297
e	0	0.0106	0	0.0045	0.0058	0.0574	0.0021	0.0316	0.0019
e_0	0	0.0187	0.1079	0.0458	0.0778	0.2858	0.002	0.0787	0.0026
u_3 a	2.1357	2.183	2.8763	2.335	2.426	5.2891	0.241	4.2683	6.905
e	1.173	1.251	1.5515	1.1141	0.5608	2.809	1.9848	2.683	1.6453
e_0	1.173	1.1923	1.8139	0.9498	0.5251	17.8089	12.540	8.1627	47.034

Table 1.

	$\sqrt{x}$ m_x=10	$\sqrt{x}$ m_x=25	log(x) m_x=10	log(x) m_x=25	e^x m_x=1	e^x m_x=5	sin(x) m_x=0	sin(x) m_x=1	sin(x) m_x=π/2
u_1 a	3.1226	3.1323	3.0868	3.1431	1.3962	1.5468	3.0262	2.9412	2.3609
e	0.0048	0.0067	0.007	0.0043	10.0755	7.077	0.0616	0.0358	0.4742
e_0	0.0048	0.0067	0.0063	0.0048	1.6407	1.3796	0.0483	0.0704	1.8262
u_2 a	3.1683	3.145	3.1468	3.1818	2.1714	2.3953	2.8294	2.746	2.281
e	0.0019	0.0007	0.0028	0.002	1.5025	0.8018	0.0661	0.0617	0.0388
e_0	0.0017	0.001	0.0022	0.0018	0.494	0.3057	0.1279	0.1794	1.3697
u_3 a	0.8596	0.0724	0.6962	0.4316	13.6229	109.918	1.1264	1.1361	1.0295
e	1.6139	1.5545	1.6189	1.6021	5.7567	6.0294	0.253	0.667	1.6542
e_0	7.0941	8.4937	8.3556	10.5984	136.883	13200	3.752	4.1863	5.8408

Table 2.

6. CONCLUSION

Today's new modeling and computing techniques make more and more necessary the development of new methods for the representation and handling of uncertainty, and with them the solving or improving the accuracy of uncertainty representation in different systems. The examinations proved that the „classical" probability theory based representation doesn't give proper results in strongly non-linear systems. Furthermore, in case of some modeling techniques the classical methods are not or only with great difficulties usable, because of the different data representations. The accurate and uniform representation of umcertainty is unsolved for different types of information loss, as well (e. g. incomplete data, estimated values).

For solving these problems some new fuzzy based uncertainty representation methods were examined. Since the communication between different systems is important, the authors also examined, how comparable, convertible are these methods to the „classical" representation. Though the examinations are far not complete, it became clear, that if we represent the uncertainty of a fuzzy number with the width of α-cut or with the integral of the membership function, we get a suitable method, which behave similar to the standard deviation and the classical and fuzzy based uncertainty measures are easily convertible. It is against the representation of uncertainty by quadratic deviation, that in this case there is not linear

relationship between the standard deviation of a random variable and the uncertainty of the corresponding fuzzy number. A great disadvantage of this method is, that the uncertainty doesn't behave according to our expectation during linear transformations.

Althought, the first results are promising, it would be deserving to examine other characteristics and to test other uncertainty representation methods, as well. Another important direction of the examination can be, how usable these uncertainty representation methods are, if we are using simplified, trapezoidal membership functions (see [8]). At present the transformations and the computing of the resultant uncertainty need too much computation, for the practical use simplified and faster methods are needed.

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