1. Historical background of fuzzy logic

The fuzzy set theory was first introduced by Lotfi Zadeh in 1965. In his seminal paper he based a new fuzzy set theorey on the ordinary or crisp set theory. The main interest of his paper was to analyse and describe complex systems. (as for example human system) At that time was the idea of multivalued logic not unknown, Jan Lukasiewicz the polnish mathematicians worked on multivalued logic at the begining of the 20-th century.
The first ten years of the fuzzy theory development were not as fast as we though. There was no special interest in fuzzy because of the general opinion that fuzzy was just a new name for probability theory. Against this opinion the development of fuzzy logic was not broken Gougen introduced L-fuzzy sets in 1967, fuzzy extensions of mathematical subjects were mentioned by Chang (1968). The relationship between many valued logic and fuzzy logic was examined by several authors that time. Approximate reasoning based on fuzzy set theory was investigated by Gougen (1968-69) and the most aspect were developed by Zadeh (1971,1972, 1975).
The first applications of fuzzy logic were made in the second part of the 70-ees?? Mamdani's steam engine (1975) showed the possibility of developement of fuzzy based controllers. On the basis of fuzzy controller theory several applications has been made for different industrial purposes such as heating exchange control (1977 Holmblad, Ostergaard), cement klin (1982 Ostergaard). Sogeno introduced his fuzzy controlled model car in 1985, Lakov his robot control application in 1985, and Nagashi, Nishizuka their traffic junction control in 1984. Risk analyses, decision making ,management have been researched by Yager, Hammerbacher, Sommer, bruce and Kandel. Applications in the field of collective decision making have been presented by Dimitrov (1983) at first.

2. Fuzzy Sets

2.1 Introduction

Sets are grouped elements of any universe of discourse. In the classical sense, sets in any universe of discourse are shared into two groups: members and nonmembers of the given universe. These type of sets are the classical or crisp sets. A set can be characterised by its elements themselves: set $B=\{1,3,5,6\}\in R$ or by its characteristic function $\varphi A= X\rightarrow [0,1]$, where $\varphi A(x)=1$ if x belongs to the set A and $\varphi A(x)=0$ if x does not belong to the set A. In many cases these type of sharp distinction is not the exact way to describe problems. For example the categories employed by humans, such as cool or warm water, hard work, tall people are classes, wich do not present a clear-cut differentiation between the elements belonging or not belonging to the set. For describing the vagueness of such classes Zadeh suggested the replacement of the charasteristic function by the so called membership function. The membership function assignes to each element in the universe of discousre a value that represents its grade of membership to the given set. These type of sets are fuzzy sets.

2.2 Basic Definitions

Membership function is the generalised form of characteristic function of sets, wich assignes a membership value to each element of a universal set (X) so called membership grade. The higher the value is, the higher is the degree of membership.
Usual forms:

$\mu_A: X\rightarrow[0,1]$ where X is the universal set, A is the fuzzy set

For example a fuzzy set of “middle aged mens” can be given by its membership function

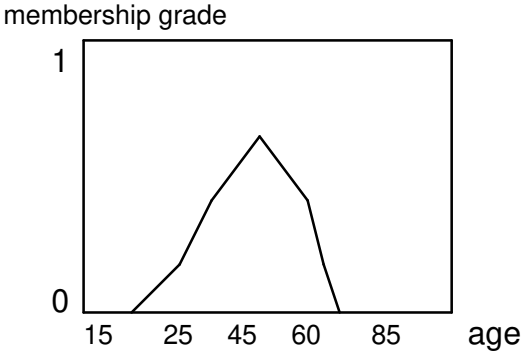


figure 1. generalised membership function

Usual membership grade function shapes:

a) Triangular shape

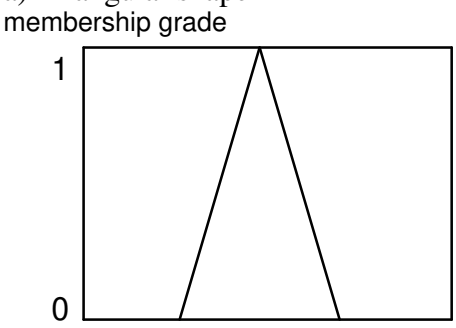


figure 2.2

b) Trapezoidal shape

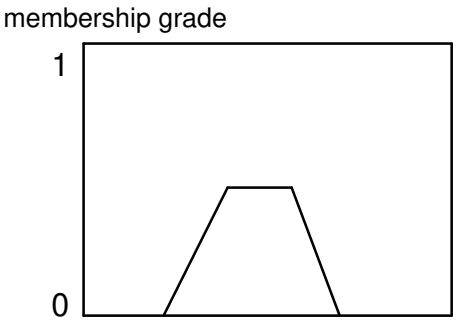


figure 2.3

b) General shape

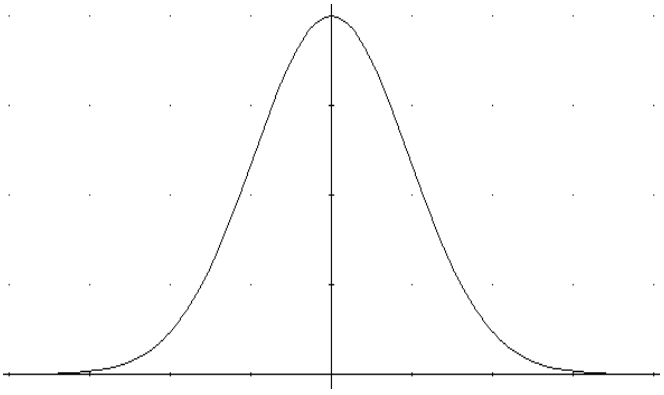


figure 2.4

A fuzzy set is given by its elements and its membership function. Formally it can be defined: a fuzzy set A in a universe of discourse X={x}, A⊆X is defined

$A=\sum \mu_i/x_i,\ x_i \in X$ (2.1)
where μi is the membership grade of xi. If X is the set of real numbers,

$A=\int_x \mu_A(x)/x$ (2.2)

Let us take a crisp universal set of temperatures for further examples given as °C values:

X={-10,-5,0,10,20,35,40,60,90,100}

For example a fuzzy set A can be given as follow:

A=0.3/-10+0.5/-5+0.8/0+0.1/20+...+μ(x)/x where μ(x) is the membership grade of x

Let us define three fuzzy sets labeled cold, comfortable, and hot temperature:

cold={1/-10,0.7/-5,0.5/0,0.1/10)
comfortable={0.3/10,1/20,0.6/35,0.2/40}
hot={0.1/40,0.5/60,0.8/90,0.9/100}

The **support** of a fuzzy set A (defined on X universal set) is the crisp set that contains all elements of X that have nonzero membership grade in A.

supp $A=\{x \in X: \mu_A(x)>0\}$
For example the support of the set *cold* is:
supp (*cold*)={-10,-5,0,10}

A **fuzzy set is empty** if it has no element with nonzero membership grade, if it has an empty support.

The **core** of a fuzzy set $\mu_A(x)= X \rightarrow [0,1]$ is a crisp subset of X, wich contains each elements with membership grade one.

$\text{core}(A)=\{x \mid \mu_A(x)=1 \text{ and } x \in X\}$ (2.3)

The **height** of a fuzzy set is the largest membership grade in the whole set. If these value is the same as the maximum possible membership grade then the fuzzy set is called normalised. If
$\mu A(x)= X \rightarrow [0,1]$ and height(A)=1 then A **normalised**. In our examples the fuzzy sets cold and confortable are normalised, the fuzzy set hot is **not normalised**.

The definition of **α-cut** is similar to the definition of the support of fuzzy sets:

$A\alpha = \{x \in X: \mu_A(x) \geq \alpha\}$ (2.4)

For example for α=0.5 the α-cut of the fuzzy set *hot* is:

$hot_{0.5}$={60,90,100}

A=B

Two fuzzy sets A and B defined on X universe of discourse are **equal** if and only if for each $x \in X$, $\mu_A(x)=\mu_B(x)$

$A \subseteq B$

A is a **fuzzy subset** in B (A is contained in B) if and only if for each $x \in X$, $\mu_A(x) \leq \mu_B(x)$

A **fuzzy number** is defined on R and it is a convex and normalised fuzzy set.

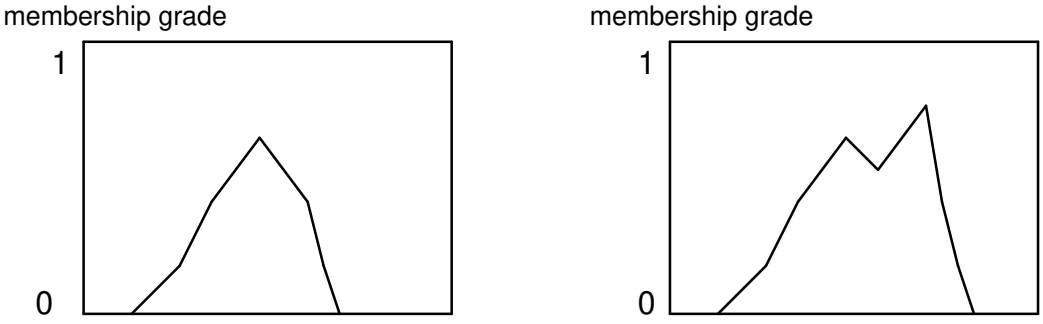


figure 2.5

2.3 Set operations introduced by Zadeh

If the range of all membership grades is the closed interval between 0 and 1 then the complement of the fuzzy set A is defined as:

$\overline{\mu_A(x)} = 1 - \mu_A(x)$ for each $x \in X$ (2.5)

For example the complement of the fuzzy set hot is given:

not *hot*={0.9/40,0.5/60,0.2/90,0.1/100}

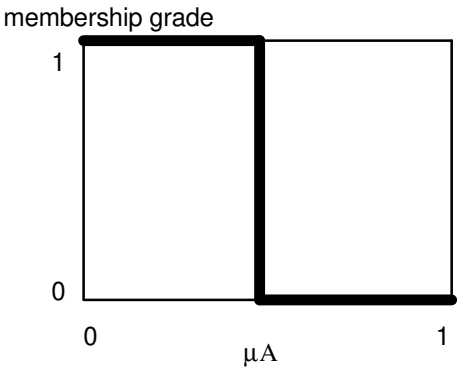


figure 2.6

The intersection of fuzzy sets A and B is defined by the membership function $\mu_{A\cap B}(x)$:

$$\mu_{A\cap B}(x)=\min [\mu_A(x), \mu_B(x)] \text{ for each } x\in X \qquad (2.6)$$

The fuzzy set $A\cap B$ is a subset of both fuzzy sets A and B. For example the intersection of fuzzy sets cold and comfortable creates the following fuzzy set:

cold$\cap$ *confortable* ={0.1/10)

The union of fuzzy sets A and B is defined by the membership function $\mu_{A\cup B}(x)$:

$$\mu_{A\cup B}(x)=\max [\mu_A(x), \mu_B(x)] \text{ for each } x\in X \qquad (2.7)$$

The fuzzy sets A and B are subsets of the fuzzy set $A\cup B$. For example the union of the fuzzy sets cold and comfortable creates the following fuzzy set:

cold$\cup$ *confortable* ={1/-10,0.7/-5,0.5/0,0.3/10,1/20,0.6/35,0.2/40)

Graphically the two operations are shown in the Fig. 2.7

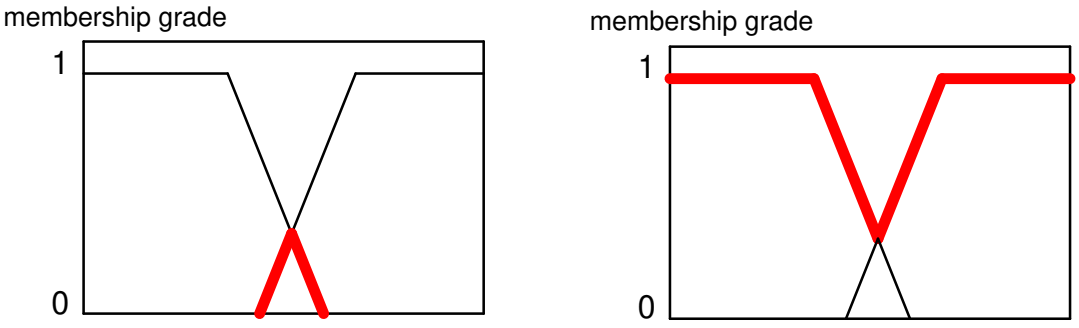


figure 2.7 Intersection and union of fuzzy sets

The **extention principle** was introduced by Zadeh in 1975 and is one of the most important tools in fuzzy sets theory. The principle addresses the problem: how can be

generalised the mapping by any f function in the crisp set X to the crisp set Y for mapping fuzzy subsets of X to Y. Let us take a fuzzy set A defined on the universe of discourse X and a function f that mappes points in the set X to the points in set Y.

$A= \mu_1/x_1+\mu_2/x_2+...+\mu_n/x_n$

$f(A)=f(\mu1/x1+\mu2/x2+...+\mu n/xn)= \mu1/f(x1)+\mu2/f(x2)+...+\mu n/f(xn)$

For the cartesian product X=X1xX2x...Xn, where A1, A2,...An are fuzzy sets defined in X1,X2,...,Xn and a crisp function f: X1xX2x...Xn$\rightarrow$Y, y=f(x1,x2,...xn), y$\in$ Y the image set B in set Y of A1,A2,...,An is defined by the membership function:

$\mu_B(y)=\max_{\{x1..xn\}\in A1..An}[\min_{i=1..n} \mu_{Ai}(xi)]$ for each y$\in$ Y, and y=f(x1,x2,...,xn)

2.4. Operations on fuzzy sets

In case of crisp set operations there are some axiomatic properties that are satisfied by all operations, let us take the crisp universal set X, and crisp sets A,B,C$\subset$X :

Boundary conditions:

$A\cup X=X$, $A\cap X=A$
$A\cup\varnothing=A$, $A\cap\varnothing=\varnothing$
for complement: $\overline{X}=\varnothing$, $\overline{\varnothing}=X$

commutativity:

$A\cap B=B\cap A$, $A\cup B=B\cup A$

associativity:

$A\cap(B\cap C)= (A\cap B)\cap C$, $A\cup(B\cup C)= (A\cup B)\cup C$

These conditions also have a plausible interpretation in interpreting a grade of membership as a grade of truth of any proposition. The boundary conditions indicate that the logical connectivities for fuzzy sets coincide with those applied in two-valued logic. The property of commutativity reflects a case in wich the truth value of a composite expression does not depend on the order of the components used in its formation.
If we accept all the above conditions, a wide variety of model classes can be defined for logical connectivities (union, intersection). These classes are called triangular norms (t-norms and co-(or s-)norms), wich generalise the intersection and union operations. A particularly interesting and active domain within the field of fuzzy set theory is the problem of specifying the union and the intersection of fuzzy subsets. In trying to select

the apptopriate fuzzy union-intersection operations for use in a given situation, consideration should be given to a number of criteria to help in this selection. In selecting the form of these operators, primary consideration must be given to the domain in wich we are using the fuzzy structure as a mathematical model. In addition , the desire to capture particular formal properties is at time a consideration. In practical case for example by a fuzzy controller the most used operators are the above shown MAX, MIN operators because of their low requriments of counting time. But for other applications as for example decision analyses where the operation time is not important, other operators can bring a better solution. Not just the norms have to satisfy any conditions but the complement as a function too.

2.5 Operations and their axiomatic propreties

2.5.1 Fuzzy complement

The complement of a fuzzy set A can be defined as a function f:[0,1]→[0,1]. These function must satisfy some requirements to be a fuzzy complement.According to the above conditions:

Axiom 1. boundary conditions: f(0)=1 and f(1)=0, as we have seen by the crisp complement

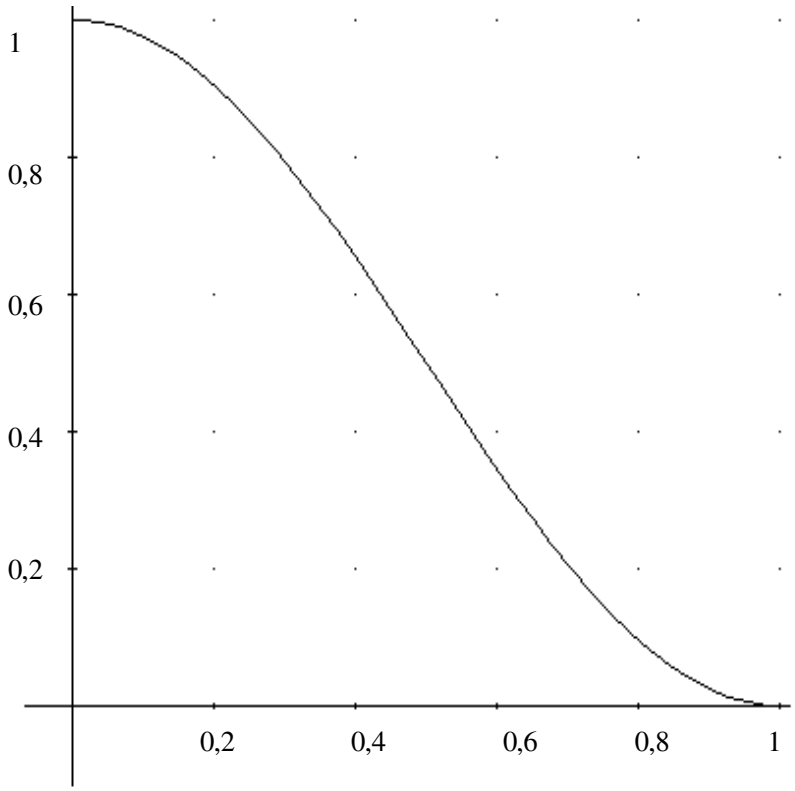
Axiom 2. monotonic nonincreasing: for any $\mu(x1),\mu(x2) \in [0,1]$, where$\mu(x1)<\mu(x2)$, $f(\mu(x1))\geq f(\mu(x2))$, where x1, x2 ∈ A fuzzy set

Axiom1 and 2 are conditions for the functions that are the most general class of fuzzy complements. Axiom 1 is necessary to satisfy the crisp boundary conditions as well. Axiom 2 is essential because we expect that the increase in the degree of membership in a fuzzy set cause a decrease in the degree of membership of its complements.
There are another from the practical wiev important and desirable requiurements for fuzzy complements:

Axiom 3. f is a continuous function

Axiom 4. f is involutive , wich states $f(f(\mu(x))= \mu(x)$ for each $\mu(x)\in [0,1]$

The last two axioms are stronger, and build a subclass of fuzzy complements. If any function is involutive, it implies its continuity.



2.8 general complement function $f(x) = \frac{1}{2}(1+\cos\pi x)$

There are many types of involutive fuzzy complements:

a) $\overline{\mu_A(x)} = 1-\mu_A(x)$ the standard complements operation

b) The Sugeno class of involutive fuzzy complements is defined by the function:

$$f_\lambda(a) = \frac{1-a}{1+\lambda \cdot a} \qquad (2.8)$$

where $\lambda \in (-1,\infty)$. Each λ value represents an involutive fuzzy complement function. For $\lambda = 0$ it becomes the crisp complement function.

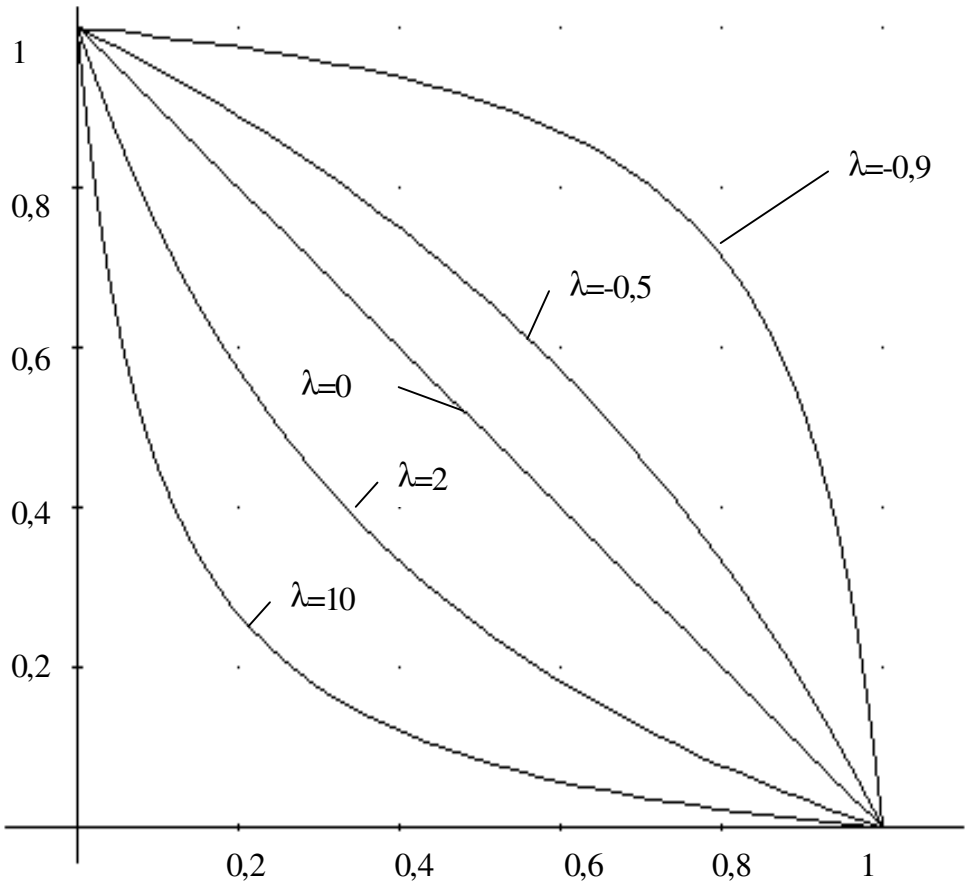


figure 2.9 The Sugeno-class of complement functions

c) The Yager class of involutive fuzzy complements is defined

$$f_w(a) = \left(1 - a^w\right)^{1/w} \qquad (2.9)$$

where $w \in (0,\infty)$. Each w value represents an involutive fuzzy complement function. For w=1` it becomes the crisp complement function.

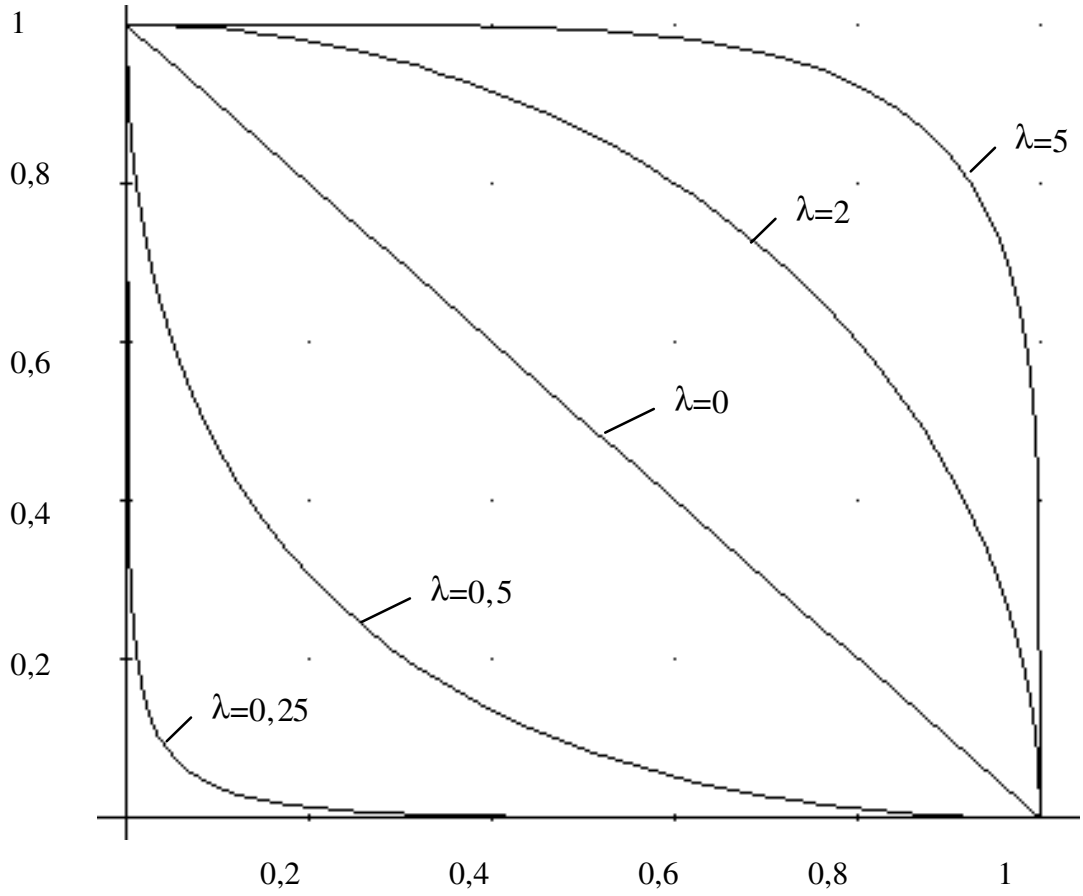


figure 2.10 The Yager-class of complement functions

Equlibrium is the value of membership degree of a fuzzy set that has the same value of complement. $f(\mu(x))= \mu(x)$. There are some rules without proof:

a) each fuzzy complement has at most one equlibrium
b) if the equlibrium of any fuzzy complement is e then

$\mu(x)\leq f(\mu(x))$ if and only if $\mu(x)\leq e$ and $\mu(x)\geq f(\mu(x))$ if and only if $\mu(x)\geq e$

c) if f continuous then it has a unique equlibrium

The dual point of a fuzzy complementer function is the point d for $\mu(x)$ represented by a real number in [0,1] and defined by:

$$f(d)-d=\mu(x)-f(\mu(x)) \qquad (2.10)$$

where f is the complement function. If f has an equlibrium, then d=e for it.

2.5.2 Fuzzy intersection, t-norm operator

An operator T :

$T:[0,1]x[[0,1]\rightarrow[0,1]$
is called t-norm.
The fuzzy intersection between A, B fuzzy sets have the following requirements and axioms:

Axiom 1. boundary conditions: $T(1,1)=1$, $T(0,1)=T(1,0)=T(0,0)=0$

Axiom2. commutativity $T(\mu(x1), \mu(x2))= T(\mu(x2), \mu(x1))$

Axiom 3. monotonic if $\mu(x1)<\mu(x2)$ and $\mu(x3)<\mu(x4)$ then $T(\mu(x1),\mu(x3))< T(\mu(x1),\mu(x3))$

Axiom 4. associative $T(T(\mu(x1),\mu(x2)),\mu(x3))=T(\mu(x1),T(\mu(x2),\mu(x3)))$

There can be other requirements for union:

Axiom 5. T is continuous

Axiom 6 T is idempotent; $T(\mu(x), \mu(x))= \mu(x)$

A fuzzy t-norm has zero divisors if there exist $x,y>0$ such that $T(x,y)=0$. A t-norm is Archimedean if $T(x,x)>x$ for every $x\in(0,1)$.

2.5.3 Fuzzy union, t-conorm operator

$S:[0,1]x[[0,1]\rightarrow[0,1]$
is called t-conorm.
The fuzzy union between A, B fuzzy sets have the following requirements and axioms:

Axiom 1. boundary conditions: $S(0,0)=0$, $S(0,1)=S(1,0)=S(1,1)=1$

Axiom2. commutativity $S(\mu(x1), \mu(x2))= S(\mu(x2), \mu(x1))$

Axiom 3. monotonic if $\mu(x1)<\mu(x2)$ and $\mu(x3)<\mu(x4)$ then $S(\mu(x1),\mu(x3))< S(\mu(x1),\mu(x3))$

Axiom 4. associative $S(S(\mu(x1),\mu(x2)),\mu(x3))=S(\mu(x1),S(\mu(x2),\mu(x3)))$

There can be other requirements for union:

Axiom 5. S is continuous

Axiom 6 S is idempotent; $S(\mu(x), \mu(x))= \mu(x)$

In general, the choice of intersection operation is not completely indepndent of the choice of union operation. This connection becomes manifest with the complement operation. The above introduction of the complement operation allows us to provide a general De Morgan's law relating t-norm and co-norms. Assume T is any arbitrary t-norm operator, then an operator S defined by

$S(a,b)=1-T(\bar{a},\bar{b})$

can be shown to be a t-conorm. More specifically, for given T operator, the corresponding S operator defined as in the above is called its dual. It can be shown that if S is a t-conorm, then $T(a,b)=1-S(\bar{a},\bar{b})$ is a t-norm. As important implication of this that if we define the union and intersection operations so that they satisfy the De Morgan's law:

$$\overline{A} \cup \overline{B} = \overline{A \cap B}$$

and

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

The following short table shows some commonly encountered t-norm and co-norm duals:

t-norm	t-conorm	called:
Min(a,b)	Max(a,b)	Min/Max
ab	a+b-ab	Product Sum
Max(0,a+b-1)	Min(1,a+b)	Bounded Sum

Some of the class that satisfy the above axioms 1..4 are shown in the table:

Union	Intersection	Range	Author
$1-Max\left[0,\left(1-a\right)^{-p}+\left(1-b\right)^{-p}-1\right]^{1/p}$	$Max\left[0,a^{-p}+b^{-p}-1\right]^{1/p}$	$p\in\left(-\infty,\infty\right)$	Schweizer & Sklar
$\frac{a+b-\left(2-\gamma\right)\cdot ab}{1-\left(1-\gamma\right)\cdot ab}$	$\frac{ab}{\gamma+\left(1-\gamma\right)\left(a+b-ab\right)}$	$\lambda\in\left(0,\infty\right)$	Hamacher
$1-\log_s\left[1+\frac{\left(S^{1-a}-1\right)\left(S^{1-b}-1\right)}{\left(S-1\right)}\right]$	$\log_s\left[1+\frac{\left(S^{a}-1\right)\left(S^{b}-1\right)}{\left(S-1\right)}\right]$	$S\in\left(0,\infty\right)$	Frank
$Min\left[1,\left(a^w+b^w\right)^{1/w}\right]$	$1-Min\left[1,\left(\left(1-a\right)^w+\left(1-b\right)^w\right)\right]^{1/w}$	$w\in\left(0,\infty\right)$	Yager
$\frac{a+b-ab-Min\left(a,b,1-\alpha\right)}{Max\left(1-a,1-b,\alpha\right)}$	$\frac{ab}{Max\left(a,b,\alpha\right)}$	$\alpha\in\left(0,1\right)$	Dubois & Prade
$\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{-\lambda}+\left(\frac{1}{b}-1\right)^{-\lambda}\right]^{-1/\lambda}}$	$\frac{1}{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{-1/\lambda}}$	$\lambda\in\left(0,\infty\right)$	Dombi

Table 2.1

2.5.4 Aggregation operators

generaly the aggregation operators are operators, wich combine several fuzzy sets to produce one fuzzy set. A mapping:

$G:[0,1]\rightarrow[0,1]$

for some $n\geq 2$ is called an aggregation operator, when it satisfy the following requirements

Axiom 1. boundary conditions $G(0,0,....,0)=0$ and $G(1,1,...,1)=1$

Axiom 2. monotonic nondecreasing $G(a_1,a_2,...,a_n)\geq G(b_1,b_2,...,b_n)$ if $a_i\geq b_i$ for all i.

Two additional axioms are usually employed, but they are not essential:

Axiom 3. G is continuous

Axiom 4. G is a symmetric function in all its arguments: $G(a_i \mid i\in N)=G(a_{p(i)} \mid i\in N)$

In the certain case the t-norms and t-conorms are aggregation operators. They were defined only for two variables but axiom 4 let us generalise them for more than two variables as well.
Averaging operators are that operators for wich:

$\min(a1,a2,...,an)\leq G(a1,a2,...an)\leq \max(a1,a2,...an)$

The averaging operator always lies between the Max and Min of the arguments. If $a^*=\max_i(a_i)$, then because of the monotonicity:

$G(a1,a2,...,an)\leq \max(a^*,a^*,...a^*)$

and since $a_i\leq a^*$. However, since from the idempotency $G(a^*,a^*,...,a^*)=a^*$, it follows that:

$G(a1,a2,...an)\leq a^*$

Similarly it can be proofed for the min operator.

There are some operators that satisfy the axioms:

The generalised mean operator that cover the whole inteval between min and max operations:

$$G_a(a_1,a_2,...,a_n)=\left(\frac{a_1^{\alpha}+a_2^{\alpha}+...+a_n^{\alpha}}{n}\right)^{1/\alpha} \qquad (2.11)$$

For fixed parameters we obtain the following aggregations operators. For $\alpha \to \infty$ this operator approaches the Max operator, $G_{\infty}=(a_1,a_2,...,a_n)=\max(a_1,a_2,...a_n)$ For $\alpha \to -\infty$ this operator approaches the Min operator, $G_{-\infty}=(a_1,a_2,...,a_n)=\min(a_1,a_2,...a_n)$

1. for $\alpha \to 0$ we obtain the geometric mean:

$$\lim_{\alpha \to 0} G_a = (a_1,a_2,...,a_n)^{1/n} \qquad (2.12)$$

2. for $\alpha=1$ we obtain the aritmetic mean:

$$G_1(a_1,a_2,...,a_n) = \frac{a_1 + a_2 + ... + a_n}{n} \qquad (2.13)$$

3. for $\alpha = -1$ we obtain the harmonic mean:

$$G_{-1}(a_1,a_2,...,a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}} \qquad (2.14)$$

Yager introduced a family of mean-like operators called the ordered weighted averaging operators. (OWA) The OWA operator of dimension is a mapping $f:R^n \to R$ that has an associated n vector W

$$w = \begin{bmatrix} w_1 \\ w_2 \\ . \\ w_n \end{bmatrix}$$

such that

1. $w_i \in (0,1)$
2. $\sum_i w_i = 1$

where $f(a_1,a_2,...a_n)=\sum_j w_j b_j$ the j^{th} largest of the a_i. Since we shall view the OWA weights as a column vector we shall find it convenient to refer to the weights with the low indicies as weights at the top and those with the higher indicies with weights at the bottom. A fundamental aspect of this operation is the reordering step, in particular , an argument ai is not associated with particular weight w_i but rather a weight is associated with particular ordered position of aggrehate. This ordering step introduces a nonlinearity into the aggregation process.

For example:

Assume:

$$w_i = \begin{bmatrix} .4 \\ .3 \\ .2 \\ .1 \end{bmatrix}$$

Let us take the aggregation f(0.8,0.1,1,0.2). In this case through reordering process we get b1=1, b2=0.8, b3=0.2, b4=0.1 Then performing the aggregation $\sum_j w_j b_j$ we get:
f(0.8,0.1,1,0.2)=(0.4)(1)+(0.3)(0.8)+(0.2)(0.2)+(0.1)(0.1)=0.69

Different OWA operators are distinguished by their weighting vector W. The most important cases are:

1.
$$W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$, in this case the aggregation $F(a_1,a_2,...a_n)=Max_i(a_i)$

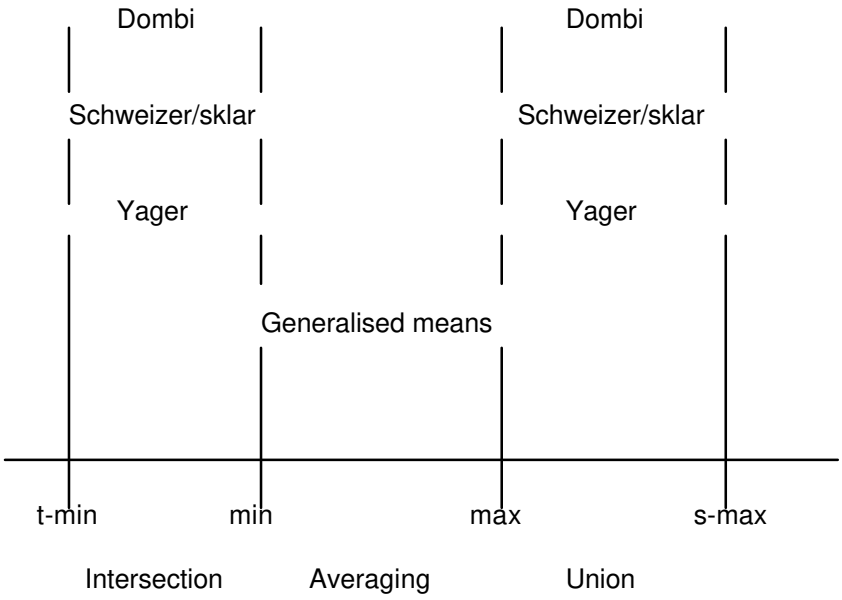
2.
$$W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$, in this case the aggregation $F(a_1,a_2,...a_n)=Min_i(a_i)$

3.
$$W = \begin{bmatrix} 1/n \\ 1/n \\ 1/n \\ 1/n \\ 1/n \end{bmatrix}$$, in this case the aggregation $F(a_1,a_2,...a_n)=1/n\sum_i a_i$

A generalised form of OWA operators is:

$$f_\alpha(a_1,a_2,...a_n;w_1,w_2,...,w_n)=\left(\sum_{j=1}^{n} w_j a_j^\alpha\right)^{1/\alpha}$$

The full scope of fuzzy aggregation operations are shown in the figure. All union and intersection operators are included and between them the generalised means.



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3. Implikation and IF-THEN rules in fuzzy logic

3.1 Crisp implication

Logic is the study of the methods of reasoning in all its possible forms. Classical logic deals with propositions that are required to be either true or false. Each proposition has an opposite, this is the negation of the proposition. The combinations of logic variables and their studies are involved in propositional logic. One of the main concerns of propositional logic is the study of rules by wich new logic variables can be produced as a function of other logic variables. A logic function assigns a truth value to a new variable for each combinations of truth values of the original variables. For n logic variables 2^{2^n} logic functions can be defined. For two input variables a and b the following functions $\varphi_1..\varphi_{16}$ are given:

a	b	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}	φ_{16}
0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Table 3.1

These simple functions are logic promitives. Any set of these logic primitives is called complete if all other logic primitives can be composed by a finite number of these primitives. The most used complete sets called logic formulas are negation, conjunction and disjunction.
The sense of Implikation is a hardly understoodable concept of logic. It was studied by greek philosophians already, and recognised as hard. The implication is a twovalued logic function and its liguistic description is

if A, then B

where A and B are logic statements and if A then B is the implication operation. Let us define the truth value function of any statement: val (),for this function
val(A), val(B)$\in$ (0,1)
How can we assign a logic function to the implication operation?

val (if A then B)=φ[val(A),val(B)]

For example for conjunction we have φ_{15}

a b or(a,b)
0 0 0

0	1	1
1	0	1
1	1	1

for disjunction φ_9:

a	b	and(a,b)
0	0	0
0	1	0
1	0	0
1	1	1

for negation:

a	non(a)
0	1
1	0

for implication is usual selected φ_{11}:

a	b	imp(a,b)
0	0	1
0	1	1
1	0	0
1	1	1

We give a simple proof for this selection:

Consider two students make an appointment:

If the weather is good then we will play soccer

similar to the above description, where the statements:

A="weather is good", B="we play soccer"

the implication is:if A then B

The question is in which case is

1. true	val (if A then B)=1
2. false	val (if A then B)=0 ?

The common sense gives us the Answer false is the implication in only one case if against the good weather they do not play soccer. val(A)=1 and val(B)=0. The implication is in the other three cases true:

1. The weather is good and they play soccer:
val(A)=1 and val(B)=1
2. The weather is not good and they do not play soccer:
val(A)=0 and val(B)=0
3. The weather is not good and they play soccer
val(A)=0 and val(B)=1

The discussion results exactly the above selected function φ_{11}

a	b	imp(a,b)
0	0	1
0	1	1
1	0	0
1	1	1

3.2 A functional interpretation of IF..THEN rules in multivalued case

Consider a black box that realises the function $f(x)=3x^{2\,i}$
It can have a graphical or lingual description as well:

if INPUT=X then OUTPUT=$3x^2$

This rule describe the function of the blackbox exactly.
There is an other situation wether we know just a partial description of a black box. This partial knowledge can be interpreted by more than one rules for example:

if X=1 then y=3
if X=3 then y=27
if X=10 then y=300
if X=1/2 then y=3/4

3.3 Axioms for implications

The axioms for implications are similar to the axioms for t-norms, s-norms and aggregations. There are fundamental axioms that should be satisfied by each implication functions. These fundamental axioms are the axom 1.-axiom 4. there are optional axioms that specialise the implications. These are axiom 5.-axiom 9.

Consider i the implication function:

$$i:[0,1]x[0,1]\rightarrow[0,1]$$

Axiom 1. for each $x\in[0,1]$:
$i(0,x)=1$
$i(1,x)=x$
$i(x,1)=1$

Axiom 2. the implication function is in his first argument comonoton, and in his second argument monoton formally:

for each $x,y,z\in[0,1]$

if $x\leq y$ then $i(x,z)\geq i(y,z)$ and $i(z,x)\leq i(z,y)$

Axiom 3. the function i satisfy the rule of kontraposition:

for each $x,y\in[0,1]$

$i(x,y)=i(i(y,0), i(x,0))$

Axiom 4. the function i staisfy the rule of premissechange:

for each $x,y,z\in[0,1]$

$i(x,i(y,z))=i(y,i(x,z))$

Axiom 5. Selfimplication:

for each $x\in[0,1]$

$i(x,x)=1$

Axiom 6.

for each $x,y\in[0,1]$

$i(x,i(y,x)=1$

Axiom 7. continuity

the function i is in [0,1] a continuous function

Axiom 8. possibility to define ordering relation by $i\in[0,1]$

for each x,y $\in$ [0,1] x$\leq$y is true if and only if i(x,y)=1

Axiom 9. the function i is involutor:

for each x $\in$ [0,1]

i(i(x,0),0)=x

3.4. Fuzzy implication forms

3.4.1 S-implications

Consider any S-norm s and a negation n. The fundament of the fuzzy implications is the twovalued imlication logic function:

i(x,y)=non(x) or y

for each x,y$\in$ {0,1}

Definition: any mapping i is an implication made by s and n for each x,y$\in$ [0,1] if

i(x,y)=s(n(x),y)

Examples for S-implications:

a) Kleene-Dienes-Implication

imp_{KD}(x,y)=max(1-x,y)

This implication is obviously an implication composed by the S-norm max(x,y) and the negation n(x)=1-x from Lukasiewicz.

b) Lukasiewicz implication

iL(x,y)=min(1,1-x+y)

This implication is obviously an implication composed by the S-norm min(1,x+y) and the negation n(x)=1-x from Lukasiewicz.

c) Implication defined by Reichenbach

iR(x,y)=1-x+x*y

This implication is obviously an implication composed by the S-norm x+y-x*y and the negation n(x)=1-x from Lukasiewicz.

d) The weak-drastic Implication

$$i_{SD} = \begin{cases} y, \text{ if } x=1 \\ 1-x, \text{ if } y=0 \\ 1, \text{ if } 0 \le x < 1 \text{ and } 0<y<1 \end{cases}$$

This implication is obviously an implication composed by the weak-drastic S-norm and the negation n(x)=1-x from Lukasiewicz. Where the weak-drastic S-norm is:

$$s_{SD} = \begin{cases} y, \text{ if } x=0 \\ x, \text{ if } y=0 \\ 1, \text{ if } x>0 \text{ and } 0<y \end{cases}$$

3.4.2 R-Implication

The fundament of the fuzzy s-implications was the boolean implication between two logic variable:

i(x,y)=s(n(x),y)

The fundament of the fuzzy r-implications is an other form of the boolean implication between two logic variable:

for each $x,y \in \{0,1\}$

$i(x,y)=\text{Max}\{z \mid (x \text{ and } z) \le y \text{ and } z \in \{0,1\}\}$

Examples:

a) Lukasievicz implication

i(x,y)=min(1,1-x+y)

is an R-implication composed by the t-norm t(x,y)=max(0,x+y-1).

b) Gödel-implication

$$i(x,y)_G = \begin{cases} 1, \text{ if } x \le y \\ y, \text{ if } x>y \end{cases}$$

is an R-implication composed by the t-norm t(x,y)=min(x,y).

c) Gougen-implication

$$i_{GO}(x,y)=\begin{cases}1, & \text{if } x=0\\ \min(1,\frac{y}{x}), & x\neq 0\end{cases}$$

3.4.3 QL-implications

This class of implication was introduced by Zadeh. The fundament of this implications is a similar form to the boolean implication:

for each $x,y\in\{0,1\}$

$i(x,y)=s(n(x),t(x,y))$

Examples:

a) Kleene-Dienes-Implication

$i_{KD}(x,y)\max(1-x,y)$

is a QL-implication composed by the t-norm t(x,y)=max(0,x+y-1), s-norm min(1,x+y), negation n(x)=1-x.

b) Zadeh-implication

$i_Z(x,y)=\max(1-x,\min(x,y))$

is a QL-implication composed by the t-norm t(x,y)=min(x,y), s-norm max(x,y), negation n(x)=1-x.

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[1]George J Klir, Tina Folger: Fuzzy Sets , Uncertainty, and Information [1988]Prentice Hall
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4.Fuzzy relations

Relations on crisp and fuzzy sets

The sense of the relation on crisp sets represents the absence of association, interaction, or interconnectness between the elements of two or more sets. The strenght of the relation among the sets is in the crisp case just by simly "yes" or "no" express. The fuzzy relations represent this strenght by membership grades, as the membership degrees are represented in fuzzy sets. The stronger the connection among the elements is the higher is the membership grade. The crisp relations can be viewed as the restricted case of the fuzzy relations where if the connection exists between elements of set the membership grade is 1, and if it does not exist the membership grade is 0.

4.1. Definitions:

Cartesian product: the cartesian product of two sets X , Y is the set of ordered pairs taken the first element from set X and the second element from the set Y. The Cartesian product is denoted by X x Y, but X x Y≠Y x X if X≠Y

$$X \times Y = ((x,y) | x \in X, y \in Y)$$

The relation among crisp sets X1,X2,X3,...,Xn is the subset of the $\times_{i \in N} X_i$ Cartesian product, wich can be defined as R(X1,X2,X3,...,Xn)

$R(X1,X2,X3,...,Xn) \subset X1 \times X2 \times X3 \times ... \times Xn$

The relation can be described by its charasteristic function, wich is two valued in the case of a crisp relation:

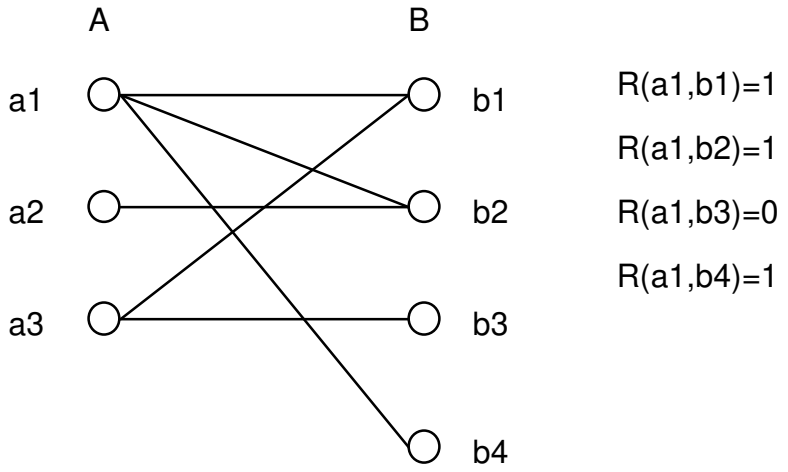
$$\mu_R(x_1, x_2, \ldots, x_n) \begin{cases} 1, & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0, & \text{if } (x_1, x_2, \ldots, x_n) \in R \end{cases} \qquad (4.1)$$

In the case of a fuzzy relation

$$\mu_R(x_1, x_2, \ldots, x_n) \in [0,1] \mid (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots X_n \qquad (4.2)$$

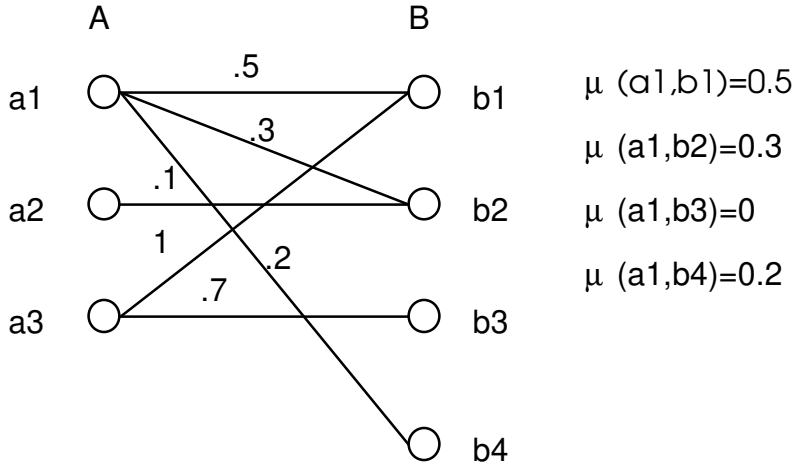
The membership function showes how strong is the relation among the elements of vector $(x_1,x_2,...,x_n)$. The fuzzy relation is defined on the Cartesian product of $X_1,X_2,....X_n$

The relation can be represented in a form of a list, a membership array, and graphical description. The following example showes these forms for crisp and fuzzy relations .



	a1	a2	a3
b1	1	0	1
b2	1	1	0
b3	0	0	1
b4	1	0	0

figure 4.1 Crisp relations



	a1	a2	a3
b1	0.5	0	1
b2	0.3	0.1	0
b3	0	0	0.7
b4	0.2	0	0

figure 4.2 Fuzzy relations

μ_R= 0.5/(a1,b1)+0.3/(a1,b2)+0.2/(a1,b4)+0.1/(a2,b2)+1.0/(a3,b1)+0.7/(a3,b3)

Subsequence:

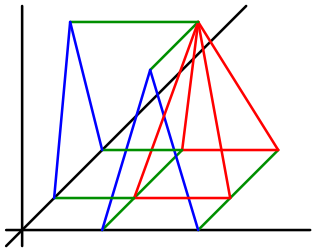
Consider $x=(xi \mid i\in N) \in \times Xi \mid i\in N$ the sequence on Cartesian product $\times Xi \mid i\in N$ and
$y=(yj \mid j\in J) \in \times Xj \mid j\in J$ the sequence on Cartesian product $\times Xj \mid j\in N$ where $J\subset N$ y is a subsequence of x if and only if xj=yj for each $j\in J$.

Formally:y<x.

Projection:

Consider the relation $R(X_1,X_2,...,X_n)$. Let us define the projection of R onto Y described by $[R\downarrow Y]$, where the variables from the set X are
$Y=(Xj \mid j\in J \subset N)$

The membership function of the relation is given as following:
$\mu_{[R\downarrow Y]}(y)=m\,ax\,\mu_R(x)$, where Y is a subsequence of X



Example: givencrisp sets $X_1=(1,2,3)$, $X2=(+,*)$, $X_3=(a,b,c)$. consider the relation $R(X_1,X_2,X_3)$ defined on Cartesian product $X=X_1\times X_2\times X_3$ { for example.: $(1,+,a)\in X$ }

$\mu_R=$ 0.8/(1,+,a)+0.6/(1,+,c)+1.0/(1,*,b)+0.7/(1,*,c)+0.2/(2,+,a)+0.4/(2,+,b)+ 0.6/(2,*,b) +0.7/(3,+,a)+0.9/(3,+,b)+0.4/(3,*,c)

Let us calculate the following relations:

a) $R_1=R\downarrow X_1\times X_2$
b) $R_2=R\downarrow X_1\times X_3$
c) $R_3=R\downarrow X_1$

From the definition of projection:
a)R_1=0.8/(1,+)+1.0/(1,*)+0.4/(2,+)+0.6/(2,*)+0.9/(3,+)+0.4/(3,*)

b) R_2=0.8/(1,a)+1.0/(1,b)+0.7/(1,c)+0.2/(2,a)+0.6/(2,b)+0.7/(3,a)+0.9/(3,b)+0.4/(3,c)

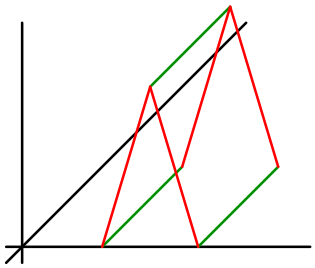
c)R_3=1.0/(1)+0.6/(2)+0.9/(3)

Cylindric extension

The cylindric extension is in some sense the inverse of the projection. Consider the set X that is a subset of Y . The elements of Y that are not contained in X are in the set Y-X . The relation R is defined on the set X The cylindric extension of the relation R into the Y-X space is denoted by $[R\uparrow Y\text{-}X]$, and its membership function is:

$$\mu_{[R\uparrow Y\text{-}X]}(y)=\mu_R(x)$$

for each $y>x$ (subsequence).



We can state that for each relation extended by the cylindric extension the projection afterwards gives back the original relation.

$$(R\uparrow(Y\text{-}X)\downarrow X)=R.$$

The inverse sequence of operation is projection and extension, but it gives back the original relation just in special cases.

Cylindric closure:

Relations can be very good approximated if they projections are known. The cylindric closure reconstruct the relation from its projections. Formally:

given: $R_1=[R\downarrow Y_1],R_2=[R\downarrow Y],\ldots,R_n=[R\downarrow Y_n]$

and considered $\times Y_i=X$

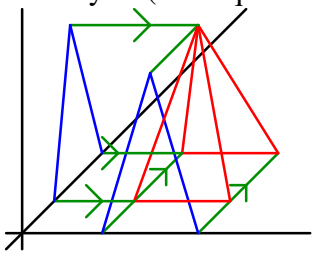
$\mathrm{cyl}\{R_i\}=R'$

where $R'\approx R$

the membership function of the cylindric extension can be calculated:

$$\mu_{R'}(x)=\min \mu_{(R_i\uparrow X-Y_i)}(x)$$

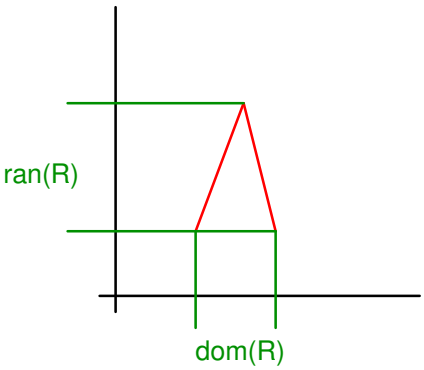
where $y<x$ (subsequence).



4.2 Binary relations

Relations defined between two set X_1, X_2 are called binary relations ($X=X_1 \times X_2$) . We denote the binary relations by the forms R(x,y) or xRy. Binary relations are described as lists arrays or graphs.

Definitions:
At first we make some definitions related to binary fuzzy relations.



Domain: the domain of any crisp relation R(X,Y) is the crisp subset of X whose members participate in the relation.:

$$dom_{R(X,Y)}=\{x \in X,(x,y) \in R \ \forall y\text{-}ra\}$$

For fuzzy relations the domain can be represented by its membership function:

$$\mu_{dom\,R}(x)=max\ \mu_R(x,y)$$

Range: the range of any R(X,Y) crisp relation is the crisp subset of Y whose members participate in the relation:

$$ran_{R(X,Y)}=\{y \in X,(x,y) \in R \ \forall x\text{-}re\}$$

For fuzzy relations the range can be represented by its membership function:

$$\mu_{ran\,R}(y)=max\ \mu_R(x,y$$

If we take a closer look at these definitions we can easily show that:

$dom(R)=[R \downarrow X]$ and $ran(R)=[R \downarrow Y]$.

Height: the height of a fuzzy relation is the largest membership grade of its membership function.
$h(R)=\max \max \mu_R(x,y)$
If the largest membership grade is 1 , $h(R)=1$ then the relation is normal, in other cases it is subnormal.

In addition to the above definotions we can build the following classes of fuzzy relations.

a) if dom (R)=X then the relation is called fully specified

b) if dom (R)$\subset$X then the relation is called not fully specified

c) if ran (R)=Y then the relation is called surjektiv

d) if ran (R)$\subset$Y then the relation is called injektiv

e) if each element in the domain of any R relation is just one times participated in the relation then the relation is called function.

f) if the point e) is not true then the relation is called multivalued relation

g) if the range of the relation involves elements that are related in the domain to more then one element then the relation is called type many to one

h) if each element of the domain is related to only one element of the range then the relation is called function with inverse.

Resolution form of binary relations: in the case of fuzzy relations we can define the α cut as we seen by the fuzzy sets. The α cut of a fuzzy relation is denoted by R_α. The resolution form of the fuzzy relation R is with the above definition of α cut:

$R= \cup \ \alpha R_\alpha \ | \alpha < h(R)$

where $\alpha R_{\alpha\ i}$ is a fuzzy relation with the membership function $\mu_{\alpha R\alpha}(x,y)= \ \alpha \ \mu_{R\alpha}(x,y)$ for each $(x,y)\in X\times Y$, and the union is the fuzzy max opration.

Let us take an example of the resolution form:
Consider Mr the matrix of fuzzy relation R:

$$M_R = \begin{vmatrix} .7 & .4 & 0 \\ .9 & 1 & .4 \\ 0 & .7 & 1 \end{vmatrix}$$

The resolution form from the definition:

$$M_R = .4\begin{vmatrix}1&1&0\\1&1&1\\0&1&1\end{vmatrix} \cup .7\begin{vmatrix}1&0&0\\1&1&0\\0&1&1\end{vmatrix} \cup .9\begin{vmatrix}0&0&0\\1&1&0\\0&0&1\end{vmatrix} \cup 1\begin{vmatrix}0&0&0\\0&1&0\\0&0&1\end{vmatrix}$$

Composition of relations:

consider two binary crisp relations P(X,Y) és Q(Y,Z) with a common set Y
The composition of these two relations is defined on the Cartesian product XxZ:

R(X,Z)=P(X,Y) o Q(Y,Z)

The properties of the binary composition are:

1) PoQ≠QoP

2) $(PoQ)^{-1}=Q^{-1}oP^{-1}$

3) (PoQ)oR=Po(QoR)

For realising a composition between fuzzy sets there are many possibily ways because of the several norms. The general form of fuzzy composition is denoted by its membership function:

$\mu_{PoQ}(x,y)= S\ \{\mu_P(x,y)\ T\ \mu_Q(y,z)\}$, where S and T are fuzzy norms

The mostly used fuzzy composition is realised by the simpliest fuzzy norms by max-min operations. The composition is called max-min composition.
The membership function of the max-min composition :

$\mu_{PoQ}(x,y)= \max\{\min[\mu_P(x,y), \mu_Q(y,z)]\}$

Let us take an example for max-min composition between two binary fuzzy relations:

$$P = \begin{vmatrix}.3&.5&.8\\0&.7&1\\.4&.6&.5\end{vmatrix} \qquad Q = \begin{vmatrix}.4&.1&.4\\.2&.9&.8\\0&1&.5\end{vmatrix}$$

$$\begin{vmatrix}.3&.5&.8\\0&.7&1\\.4&.6&.5\end{vmatrix} o \begin{vmatrix}.4&.1&.4\\.2&.9&.8\\0&1&.5\end{vmatrix} = \begin{vmatrix}.3&.5&.5\\.2&1&.7\\.4&.6&.6\end{vmatrix}$$

Where we calculated the first column as follows:

0.3=max{min(0.3,0.4),min(0.5,0.2),min(0.8,0.0)}
0.2=max{min(0.0,0.4),min(0.7,0.2),min(1.0,0.0)}
0.4=max{min(0.4,0.4),min(0.6,0.2),min(0.5,0.0)}

An alternative form of fuzzy binary compositions is the max-product composition, where as t-norm we take the algebraic product. The membership function of the max-product composition is:

$\mu_{PoQ}(x,y) = \max \{\mu_P(x,y) * \mu_Q(y,z)\}$

The max-product composition gives the following result for the above P and Q relations.

$$\begin{vmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{vmatrix} \text{o} \begin{vmatrix} .4 & .1 & .4 \\ .2 & .9 & .8 \\ 0 & 1 & .5 \end{vmatrix} = \begin{vmatrix} .12 & .8 & .4 \\ .14 & 1 & .56 \\ .16 & .54 & .48 \end{vmatrix}$$

Relational join:

Consider two binary relations:Adott két bináris reláció:
$P \subset X \times Y$, $Q \subset Y \times Z$
The relational join $R=P*Q$ for each $(x,y,z) \in R$ and consider $(x,y) \in P$, $(y,z) \in Q$
is denoted by the membership function:

$\mu_R(x,y,z) = \min \{\mu_P(x,y), \mu_Q(y,z)\}$, for each $x \in X$, $y \in Y$, $z \in Z$

The relational join is connected to the composition. By a simply modification of the relational join we obtain the composition of two relations.

$\mu_{PoQ}(x,y) = \max \{\mu_{PoQ}(x,y,z)\}$ for each $x \in X$, $z \in Z$

Note the inverse operation is not possibile.

Example:

Given sets X=(x1,x1), Y=(y1,y2), Z=(z1,z2)
P and Q are fuzzy relations defined on (X,Y)

P=0.3/(x1,y1)+0.8/(x1,y2)+0.6/(x2,y1)+1.0/(x2,y2)
Q=0.4/(y1,z1)+0.9/(y1,z2)+0.0/(y2,z1)+0.2/(y2,z2)

The relational join of these relations is:

P*Q=0.3/(x1,y1,z1)+0.3/(x1,y1,z2)+0.0/(x1,y2,z1)+0.2/(x1,y2,z2)+0.4/(x2,y1,z1)+0.6/(x2,y1,z2)+0.0/(x2,y2,z1)+0.2/(x2,y2,z2)

As before mentioned the max-min composition can be simply obtained:

PoQ=0.3/(x1,z1)+0.3/(x1,z2)+0.4/(x2,z1)+0.6/(x2,z2)

4.3. Binary relations on same sets

As it is possible to define binary relations between elements of different sets, it is possible between elements of the same set. This relation can be represented the same way as mentioned before by general binary relations. Binary relations are described as lists arrays or graphs.

Binary crisp relations have three main properties that give the possibility to distinguish them:

a) the relation R(X,X) is reflexive if and only if $(x,x)\in R$ for each $x\in X$. If there is any $x\in X$ that it is not true then the relation is irreflexive. If $(x,x)\notin R$ for each $x\in X$-re then the relation is antireflexive.

b) the relation R(X,X) is symmetric if and only if $(x,y)\in R$ for each $x,y\in X$. If there is any $x,y\in X$ that it is not true then the relation is aymmetric. If $(x,y)\in R$ and $(y,x)\in R$ is true only if $x=y$ for each $x,y\in X$ then the relation is antisymmetric. If either $(x,y)\in R$ or $(y,x)\in R$ whenever $x\neq y$ then the relation is strictly antisymmetric.

c) the relation R(X,X) is transitive if and only if $(x,z)\in R$ whenever both $(x,y)\in R$ and $(y,z)\in R$ for at least one $y\in X$. If it is not true then the relation is nontransitive. If $(x,z)\notin R$ whenever $(x,y)\in R$ $(y,z)\in R$ then the relation is antitranzitive.

The same properties are expressed for fuzzy relations by their membership functions:

a) the fuzzy relation R(X,X) is reflexive if and only if

$$\mu_R(x,x)= 1$$

for each $x\in X$. If there is any $x\in X$ that it is not true then the relation is irreflexive. If $(x,x)\notin R$ for each $x\in X$-re then the relation is antireflexive.

b) the fuzzy relation R(X,X) is symmetric if and only if

$$\mu_R(x,y)=\mu_R(x,y)$$

for each $x,y\in X$. If there is any $x,y\in X$ that it is not true then the relation is aymmetric. If $\mu_R(x,y)>0$ and $\mu_R(y,x)>0$ is true only if $x=y$ for each $x,y\in X$ then the relation is antisymmetric.

c) the fuzzy relation R(X,X) is transitive (exactly max-min tranzitive) if and only if

$$\mu_R(x,z)\geq \max \min [\mu_R(x,y),\mu_R(y,z)]$$

for each (x,z)∈ X2. If it is not true then the relation is nontransitive.

Tranzitive closure: the transitive closure denoted by R_T of a fuzzy relation R(X,X) is transitive, it involves the fuzzy relation R(X,X) and has the least membership grade. The transitive closure can be generated by the following algorythm

a) R′= R ∪ (RoR)

b) ha R′≠ R, akkor R=R′ GOTO (a)

c) R′=R_T

If the used composition operation is a max-min composition then R_T is called max-min transitive closure. Consider the following fuzzy relation R given by its matrix form:

$$M_R = \begin{vmatrix} .7 & .5 & 0 \\ 0 & 0 & 1 \\ 0 & .3 & .8 \end{vmatrix}$$

Let us calculate the transitive closure of the relation:

$$M_{RoR} = \begin{vmatrix} .7 & .5 & .5 \\ 0 & .3 & .8 \\ 0 & .3 & .8 \end{vmatrix} \quad M_{R\cup(RoR)} = \begin{vmatrix} .7 & .5 & .5 \\ 0 & .3 & 1 \\ 0 & .3 & .8 \end{vmatrix}$$

$$M_{RoR} = \begin{vmatrix} .7 & .5 & .5 \\ 0 & .3 & .8 \\ 0 & .3 & .8 \end{vmatrix} \quad M_{R\cup(RoR)} = \begin{vmatrix} .7 & .5 & .5 \\ 0 & .3 & 1 \\ 0 & .3 & .8 \end{vmatrix}$$

4.4. Classification of relations based on their properties

a) Equivalence and similarity relations

The crisp relation that satisfy reflexivity, symmetry and transitivity is called equialence relation. The binary fuzzy relation that satisfy the same properties is called similarity relation. (the transitivity is in this case max-min transitivity) The similarity relation builds the group of elements called similarity class whose elements have membership grades depending on their similarity to a defined $x \in X$. If the membership grades of smilarity are restricted to the values 0 and 1 then we obtain the equialence class

For example the mentioned α-cuts of any R relation (R_α) are equialence relations

$R = \cup \ \alpha R_\alpha \ | \ \alpha < h(R)$.

b) Compatibility and tolerance relations

The binary relations that satisfy the properties reflexivity and symmetry are called compatibility or tolerance relations.

c) Orderings:

The symmetry was the most important property in the case of similarity and tolerance relations. The orderings are characterised the same way by the asymmetry.
Consider a crisp binary relation R(X,X) that is reflexive, asymmetric and tranzitive ,it is called partial ordering. Formally: $x \leq y$
The inverse of the partial ordering is $R^{-1}(X,X)$: $x \geq y$. The fuzzy partial ordering is characterised by reflexivity , asymmetry, and any form of fuzzy tranzitivity.

To summarise the properties and the by them defined classes of fuzzy relations we show them in a common table:

	reflexive	antireflexive	symmetric	antisymmetric	tranzitive
similarity	•		•		•
quasi-equivalence			•		•
tolerance	•		•		
partial ordering	•			•	•
quasi -ordering	•				•
strictly ordering		•		•	•

4.5 An other type of fuzzy relations : fuzzy preference relations and its application in preference modelling

These unit showes an axiomatic approach to the definition of fuzzy strict preference P, indifference I and incomparability J with fuzzy preference relations.

Preference modelling is an indispensable part of operational research, decision making, sociology, economy etc. The classical theory of preference modelling is adequate in a lot of situations. If A is a crisp set of alternatives and R is a binary crisp relation on crisp set A with

aRb $\Leftrightarrow$ a is not worse than b,

where a,b $\in$ A, then one can define three binary relations associated with R in the following way:
1) strict preference relation P: aPb $\Leftrightarrow$ aRb and not bRa

2) indifference relation I: aIb $\Leftrightarrow$ aRb and bRa

3) incomparability relation J: aJb $\Leftrightarrow$ not aRb and not bRa

The set theoretical equialent of the verbal expressions are:

$P=R\cap R^d$ (4.5.1)

$I=R\cap R^{-1}$ (4.5.2)

$P=R^c\cap R^d$ (4.5.3)

where R^{-1} ,$R^d$and R^c are the inverse, the complement and the dual of R ($R^d = (R^{-1})^c$), respectively. On the other hand, P is asymmetric, I and J are symmetric relations.
Further basic connections among P,I,J and R can be considered:

$P\cup I=R$ (4.5.4)
$P\cap I=\varnothing$ (4.5.5)
$P\cap J=\varnothing$ (4.5.6)
$I\cap J=\varnothing$ (4.5.7)

Its easy to see that relations (4.5.1), (4.5.2), (4.5.3)have equivalent forms in terms of the set-difference operation (\) as follow:

$P=R\backslash R^{-1}$ (4.5.8)

$I=R\backslash R^d$ (4.5.9)

$J=R^c\backslash R^{-1}$ (4.5.10)

Now we have to examine the strict preference, indifference, incomparability when the R relation is a fuzzy preference relation.. Basic properties for the exemination are the classical forms (4.5.4)-(4.5.7).

Axioms for P,I,J:

We assume that R is a fuzzy preference relation, i.e. a function R: AxA$\rightarrow$[0,1] such that for any a,b$\in$ A, R(a,b) is the truth value of the statement " a is not worse than b".

Axiom 1.
For any two alternatives a,b the values of P(a,b), I(a,b) and J(a,b) depend only on R(a,b) and R(b,a).

According to Axiom 1 wich is called 'independence of irrelevant alternatives', there exists three functions p,i,j :[0,1]x[0,1]$\rightarrow$[0,1] such that

$$P(a,b)=p(R(a,b),R(b,a)) \qquad (4.5.11)$$

$$I(a,b)=i(R(a,b),R(b,a)) \qquad (4.5.12)$$

$$J(a,b)=j(R(a,b),R(b,a)) \qquad (4.5.13)$$

Axiom 2.

p(x,y) is nondecreasing in its first place and nonincreasing in its second place;
i(x,y) is nonincreasing with respect to both arguments;
j(x,y) is nonincreasing in each one of its places

Axiom 2. is called a positive association principle.

Axiom 3.
P is asymmetric, I and J are symmetric relations.

Axiom 3 asserts that
a) min{P(a,b), P(b,a)}=0 for any a,b$\in$ A
b) I(a,b)=I(b,a) for every a,b$\in$ A
b) J(a,b)=J(b,a) for any a,b$\in$ A

or equivalently,
a')x$\leq$y implies p(x,y)=0 (x,y$\in$ [0,1])
b')i(x,y)=i(y,x) for every x,y$\in$ [0,1])
c')j(x,y)=j(y,x) for every x,y$\in$ [0,1])

Now we need approriate functions for p, i and j in accordance with Axiom1..Axiom 3. and also with fuzzy set theoretic operations.

Let us take a t-norm T: $[0,1]x[0,1]\rightarrow[0,1]$, that is commutative, associative, nondecreasing, continuous, Archimedean and has zero divisors. (see in unit 2.5.2) For any t-norm T a pseudocomplement $T^{\rightarrow}$ can be defined by:

$$T^{\rightarrow}(x,y)=\sup\{z;T(x,z)\leq y\}. \qquad (4.5.14)$$

$T^{\rightarrow}(x,y)$ can be viewed as a fuzzy implication operator.

Lemma (proof see in [1])

a) for any continuous t-norm T we have $T(x,y)\leq z$ if and only if $T^{\rightarrow}(x,z)\geq y$.

b)for any continuous t-norm T the equation $T(x, T^{\rightarrow}(x,y))=y$ holds if and only if $x\geq y$.

c) if T is continuous and has zero divisors, then T is Archimedean.

d) if T is a continuous, Archimedean t-norm with zero divisors then t is strictly increasing in both arguments on the set $\{(x,y);T(x,y)>0\}$.

Assume now that (T,S,n) is a De Morgan triple, i.e. T is a continuous t-norm, n is a strict negation and

$$S(x,y)=n^{-1}[T(n(x),n(y))] \qquad (4.5.15)$$

is the n-dual t-conorm to T.
Denoting x=R(a,b), y=R(b,a) for short and using (4.5.11), (4.5.12), (4.5.13) one can easily translate relations (4.5.4), (4.5.5), (4.5.6) and (4.5.7).

$$S(p(x,y),i(x,y))=x \qquad (4.5.16)$$

$$T(p(x,y),i(x,y))=0 \qquad (4.5.17)$$

$$T(p(x,y),j(x,y))=x \qquad (4.5.18)$$

$$T(i(x,y),j(x,y))=x \qquad (4.5.19)$$

If we define P, I, J by (4.5.1), (4.5.2) and (4.5.3) the we obtain

$$p(x,y)=T(x,n(y)) \qquad (4.5.20)$$

$$i(x,y)=T(x,y) \qquad (4.5.21)$$

$$j(x,y)=T(n(x),n(y)) \qquad (4.5.22)$$

Now we try to use (4.5.8), (4.5.9) and (4.5.10) we get:

$$i(x,y)=p(x,n(y)) \qquad (4.5.23)$$

Theorems (proof see by Fodor [1]):

4.5.1) There is no De Morgan triple (T,S,n) such that (4.5.16) and (4.5.17) hold for p and i defined by (4.5.20) and (4.5.21) respectively.

4.5.2) There is no De Morgan triple (T,S,n) such that p and i defined by (4.5.23) fulfil equation (4.5.16)

One way to get a solution for p, i, j change or leave out some of the formulare (4.5.1)-(4.5.3) and/or (4.5.8)-(4.5.10).
Equation (4.5.4) has an equivalent form in the crisp case :

$R^d = P \cup I$,
wich means in our situation the following:

$$S(p(x,y),j(x,y))=n(y) \qquad (4.5.24)$$

The system consists of equations (4.5.16) and (4.5..1), without any assumption on the form of p, i, j. Its solution is given in the next theorem.

Theorem 4.5.3:
Assume that Axiom 1...,Axiom 3. hold. Then

<p, i, j, T, S, n>

fulfils the equations

$$S(p(x,y),i(x,y))=x$$

$$S(p(x,y),j(x,y))=n(y)$$

if and only if
a) T is continuous, Archimedean t-norm with zero divisors

b) $n(x)= T^{\rightarrow}(x,0)$

c) $S(x,y)=n[T(n(x),n(y))]$

d) $p(x,y)=T(x,n(y))$

e) $i(x,y)=\min\{x,y\}$

f) $j(x,y)=\min\{n(x),n(y)\}$

Summarising the results reasonable expression for P, I, and J are proposed as follows:

$$P(a,b)=T(R(a,b),n[R(b,a)])$$

$$I(a,b)=\min\{R(a,b),R(b,a)\},$$

J(a,b)=min{n[R(a,b)],n[R(b,a)]},

where (T,S,n) is given e. g. by Theorem 4.5.3

The result of our investigation can be very useful in decision supportter systems, and has practical applications.

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