1. Fundamental of fuzzy sets and logic

Introduction to fuzzy logic control

1.1 Historical background

Historical Development: Fuzzy System

The concept of fuzzy set was suggested first in the mid sixties by L.A.Zadeh to achieve a simplified modelling of complex system.The publication that appeared until this publication were dedicated mostly to a theoretical analysis of promising application of fuzzy sets, which on the whole resulted from fuzzyfication of the known theories. A thoroughly commented bibliography that covers the first decade of this domain of research was written byB.R. Gaines and L.J. Kohout [Gaines77]. As representative of the publication that appeared in the domain of fuzzy sets and fuzzy systems up to 1978 the book [Dubois80] by D.Dubois and H. Prade should be mentioned.

The most important treatises of Saudi, who has influenced the development of the whole theory up to now in an authoritative way, are collected in [Yager87]. In [Dubois93] important papers are compiled and discussed.

Since about 1975, besides a further theoretical deepening, there were attempts to develop applications for laboratories as well as large scale technical utilisation. As a pioneer in the domain of fuzzy control E.H. Mamdani [Mamdani76] is well respected. His control syst4m for a cement kiln [Holmblad82] was the first large scale application that was realised successfully. In Japan the possibilities opened up by fuzzy control were recognized and promoted very early. This not only led to spectacular application like the underground in Sendai, but also to more simple realization in consumer goods, that were extremely successful [Terano91]. An additional field for employing fuzzy methods was seen in the fields of knowledge processing, artificial intelligence [Smets88] and analyse of fuzzy data [Miyamoto90, Bezde81, Bandemer92].In Europe and the U.S.A. Zadeh’s basic concepts and even same first results in control engineering and mathematics met with only modest interest. R.E.Kalman remarked in 1972 at a presentation given by Zadeh:

'His proposal could be severely, ferociously even brutally criticised from a technical point of view. Professor Zadeh's fears of unjust criticism can be mitigated by recalling that the alchemists were not prosecuted for their beliefs but because they failed to produce gold."

1.2 Fuzzy sets and Logic

Introduction:
Science consist in the use of modelling and experimentation of natural and artificial phenomena in human terms. A model is a system of relation between data; some of these data are certain, other are random, other are uncertain. Models with certain or/and random data are well known and have been used in for a long time. When scientists found uncertain data they generally tried to transform uncertainty to randomness using a prior law of probability such as the uniform Laplace law (law of equiprobability), or sometimes using a Bayesian process.
The difference between randomness and uncertainty: Uncertainty and randomness are epistemologically different: a random datum must be measurable and objective, and uncertain datum is not measurable and is subjective-uncertainty and randomness must be considered separately.
!!!Be careful: probabilities uncertainty is not equivalent with probabilistic uncertainty.

SET THEORY-Classical and Fuzzy Sets
Conceptually, a fuzzy set can be defined as a collection of elements in a univers of information where the boundary of the set contained in the univers is ambiguous, vague, and otherwise-fuzzy. It is instructive to introduce fuzzy sets by first reviewing the elements of classical (crisp) set theory.

1.2.1 Classical Sets

Define X as the set of all objects with the same characteristics, called the univers of discourse, whose individual elements are denoted by x. The elements of the univers can be discrete and finit or continous and infinite. also, define the cardinality number, n_X , as the total numbers of elements in X. A set a consists of collections of some elements in X; set A is subset of the univers X. Furthermore, the ffollowing notation holds:

$$x \in X \rightarrow \text{x belongs to X}$$

$$x \in A \rightarrow \text{x belongs to A}$$

$$x \notin A \rightarrow \text{x does not belongs to A}$$

For sets A and B on X, we also have

$$A \subset B \rightarrow \text{A is contained in B, } \forall x \in A, \text{ then x} \in B$$

$$A \subseteq B \rightarrow \text{A is fully contained in B}$$

$$(A = B) \rightarrow A \subseteq B \text{ and } B \subseteq A$$

We define the null set,∅, as the set containing no elements., and the whole set, X as the set of all elements in the univers. all subsets of X comprise a special set called the power set, P(X). for example, for the following univers x, the power set is enumerated.

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Example: We have the univers comprised of three elements, X={a,b,c}, so the cardinality is n_x *=3. the power set is ;*

P(X)={∅, {a},{b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}

The cardinality of the power set, denoted $n_{P(X)}$*, is found as:*

$$n_{P(X)} = 2^{n_x} = 2^3 = 8$$

===

Note that if the cardinality of the univers is infinite, then the cardinality of the power set is also equal to infinity, i.e., If $n_X = \infty$ then $n_{P(X)} = \infty$.

1.2.1.1 Operation on Classical Sets

Let a and B be two sets in the univers X. The union between the two sets , denoted $A \cup B$, represents all those elements in the univers which reside in (or belong to0 either the set a or the set b or both set a and B. The intersection of the two sets, denoted $A \cap B$, represents all those elements in the univers x which reside in (or belong to0 both sets A and B, simultaneousily. the complement of a set A, denoted $\bar{A}$, is defined as the collection of all elements in the univers which do not reside in the set A. the difference of a set a with respect to b, denoted A\B, is defined as the collection of all elements in the univers which reside in A and which do not reside in B simultaneously. These operation are shown below:

Union:

$$A \cup B = \{x | x \in A \text{ or } \text{x} \in B\}$$

Intersection:

$$A \cap B = \{x | x \in A \text{ and } \text{x} \in B\}$$

Complement:

$$\bar{A} = \{x | x \notin A \text{ or } \text{x} \in X\}$$

Difference:

$$A \backslash B = \{x | x \in A \text{ (or } \text{x} \notin B) \}$$

These operations are shown below in terms of Venn diagrams in figures 2-1 through 2-4

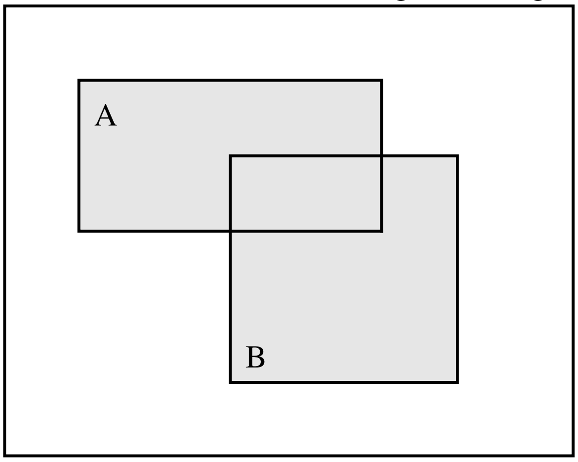

Figure 2-1. Union of sets A and B.

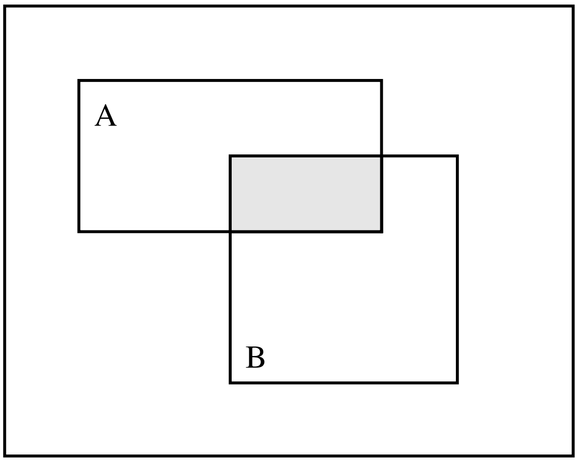

Figure 2-2.Intersection of sets A and B.

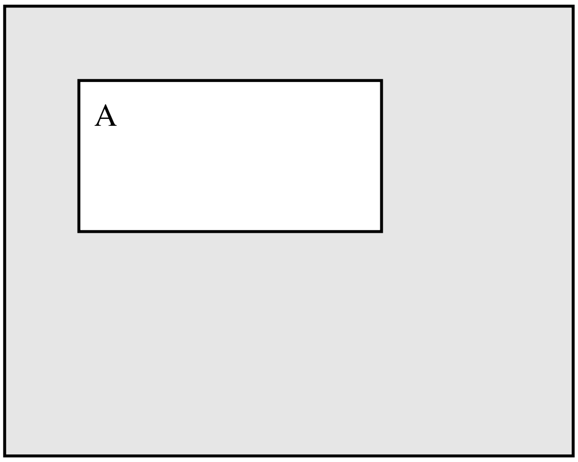

Figure 2-3. Complement of set A.

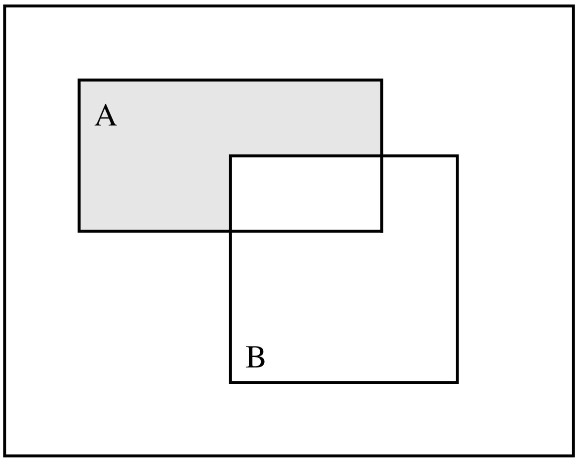

Figure 2-4. difference operation A\B

1.2.1.2 Properties of classical (Crisp) sets

Certain properties of sets are important to consider because of their influence on the mathematical manipulation of sets. The most appropiate properties for purposes of defining classical sets, to show their similarity to fuzzy sets, are:

Commutivity:

$$A \cup B = B \cup A$$

$$A \cap B = (B \cap A)$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = ((A \cap B) \cup (A \cap C))$$

Idempotency:

$$A \cup A = A$$

$$A \cap A = A$$

Identity:

$$A \cup \varnothing = A$$

$$A \cap \varnothing = \varnothing$$

$$A \cap X = A$$

$$A \cup X = X$$

Transitivity:

$$\text{If } A \subseteq B \subseteq C, \text{ Then } A \subseteq C$$

Involution:

$$\overline{(\bar{A})} = A$$

Two special properties for set operations are known as the Excluded Middle Laws and De Morgan's Laws. The excluded middle laws are very important properties because these are the only set operations described here that are not valid for both classical sets and fuzzy sets, while DeMorgan,s law are important because of there use fulness in proving tautologies and contradictions in logic, as well as their utility in a host of other setset operations and proofs. the excluded middle laws are actually two laws: the first called the Law of Excluded Middle, deals with the union of a set a and its complement, and the second, called the Law of Contradiction, represents the intersection of a set a and its complement. These laws are enumerated below for two sets A and B, and DeMorgan's laws are displayed in the shaded areas of the Venn diagrams in Figures 2-5 and 2-6.

Excluded Middle Laws:

1) Law of Excluded Middle

$$A \cup \bar{A} = X$$

2) Law of Contradiction

$$A \cap \bar{A} = \varnothing$$

DeMorgan's Laws:

$$(\overline{A \cap A}) = (\bar{A} \cup \bar{B})$$

$$(\overline{A \cup A}) = \ (\bar{A} \cap \bar{B})$$

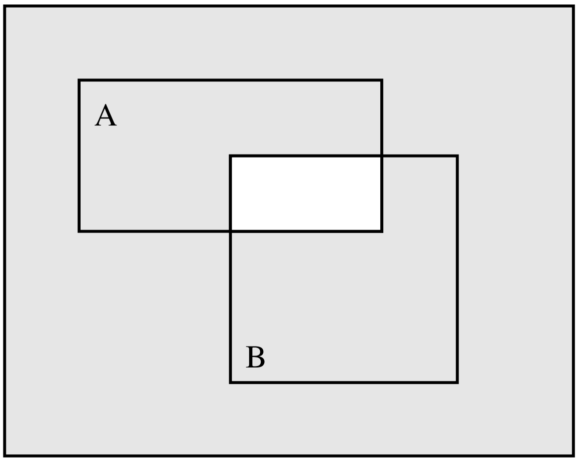

Figure 2-5. First DeMorgans Law.

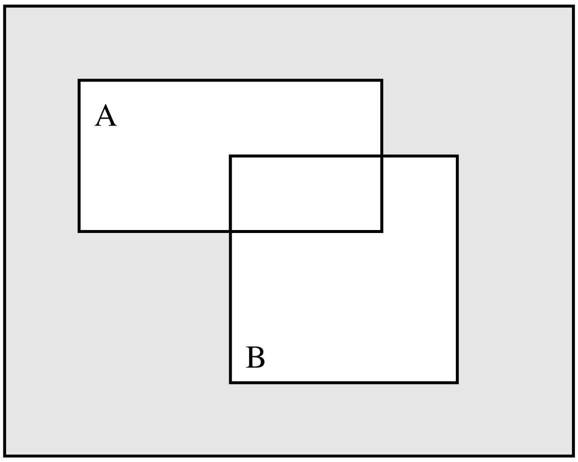

Figure 2-6. second DeMorgan Law.

1.2.1.3 Mapping of Classical Sets to Functions

Mapping is an important concept in relating set-theoretic forms to function-theoretic representation of information. mapping in its most general form can be used to map elements and subsets one one univers of discours to elements or sets in another univers. Suppose x and y are two different univers of discourse (information). if an element x is contained in X and it corresponds to an element y contained in Y, this is generally termed a mapping from X to y, or f:X->Y. as a mapping, the caracteristic (indicator) function χ_A is defined by

$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

where χ_A expresses “membership’ in set a for the element x in the universe. this membership idea is a mapping , from an element x in univers X to one of the two elements in univers Y; i.e., to the element 0 or 1.

For any set A defined on the univers X, There exist a value-set, V(A) under the mapping of the caracteristic function , χ . By convention the null set $\varnothing$ is assigned the membership value 0 and the whole set X is assigned the membership value 1.

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Example:Continuing with the previous example of the univers with three elements, X={a,b,c}, we desire to map the elements of power set of X, i.e., P(X) to a univers , Y, consisting in only two elelments (the caracteristic function).

==Y

Y={0,1}

as before, the elements of the power set are enumerated below,

P(X)={$\varnothing$, {a},{b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}

And now the elements in the value set V(A) as determined from the mapping are,

V[P(X)]= [{0,0,0}, {1,0,0}, {0,1,0], {0,0,1}, { 1,1,0}, {0,1,1}, {!,0,1}, {1,1,1}]

==

Now, define two sets on the univers X, sets Aand b. the union of these two sets in terns of function-theoretic form s is given by,

Union:

$$A \cap B \rightarrow \chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x) = max(\chi_A(x), \chi_B(x))$$

The intersection of these two sets in function-theoretic terms is given by, Intersection:

$$A \cap B \rightarrow \chi_{A \cap B}(x) = \chi_A((x) \wedge \chi_B(x) = min(\chi_A(x), \chi_B(x)))$$

The complement of a single set on universe, say A is given by,
Complement:

$$\bar{A} \Im c_{\bar{A}}(x) = (1 - c_A(x))$$

For two set on the same univers, say A and B, if one set (A) is contained in another set (B), then
Containment:

$$A \subseteq B \rightarrow \chi_A(x) \leq \chi_B(x)$$

1.2.2 Fuzzy Sets

The theory of fuzzy sets deals with unprobabilistic uncertainty. In many situations in modelling it is much better to use fuzzy set theory rather than probability theory because a more realistic model results. If same data in the model are not measurable, foe one reason or another then valuation (which is the procedure for fuzzy theory) is recommended. Many people confuse the different theories, or think that what one can do with fuzzy sets one can also do in probability theory: ‘Anything you can do [with fuzzy sets] I can do [with probability] better” [Lindley87].
This is grave error; the theories are fundamentally different. now let us define the concept of fuzzy sets.
Definition : A fuzzy set μ of X is a function from the reference set X to the unit interval, i.e.

$$\mu : X \rightarrow [0, 1]$$

F(X) denotes the set of all fuzzy sets of X.

Each ordinary set A (in the context of fuzzy systems sometimes called crisp set) can be interpreted (via the caracteristic function) as a special kind of fuzzy set. !!!
In classical sets, or crisp sets, the transition between membership and non-membership in a given set for an element in the univers is abrupt and well-defined (said to be “crisp”). For an element in a univers which contains fuzzy sets this transition can be gradual. This transition among various degrees of membership can be thought of as conforming to the fact that the boundaries of the fuzzy sets are vague and ambiguous. Hence membership of an element from the univers in this set is measured by function which attempts to describe vagueness and ambiguity.
A fuzzy set then is a set containing elements which have varying degrees of membership in the set. This idea is contrasted with classical, or crisp, sets becouse members of a crisp sets would not not be members unless their membership was full or complete in that set (i.e., their membership is assigned a value of 1). Elements in a fuzzy set, becouse their membership can be a value other than complete, can also be members of other fuzzy sets on the same univers. Elements of fuzzy sets are mapped to univers of “membership values” using a function-theoretic form. Fuzzy sets are denoted by a set symbol with a tilde understrike; so for example, $\underset{\sim}{A}$ would be the “fuzzy set” A. This function maps elements of a fuzzy set $\underset{\sim}{A}$ to a real numbered value on the interval 0 to 1. If an element in the univers, say x, is a member of fuzzy set $\underset{\sim}{A}$ then this mapping is given as,

$$\mu_{\underset{\sim}{A}}(x) \in [0, 1]$$

$$\underset{\sim}{A} = \left(x, \mu_{\underset{\sim}{A}}(x) \middle| x \in X \right)$$

These mapping are shown below in Figures 2-7 and 2-8 for crisp and fuzzy sets, respectively:

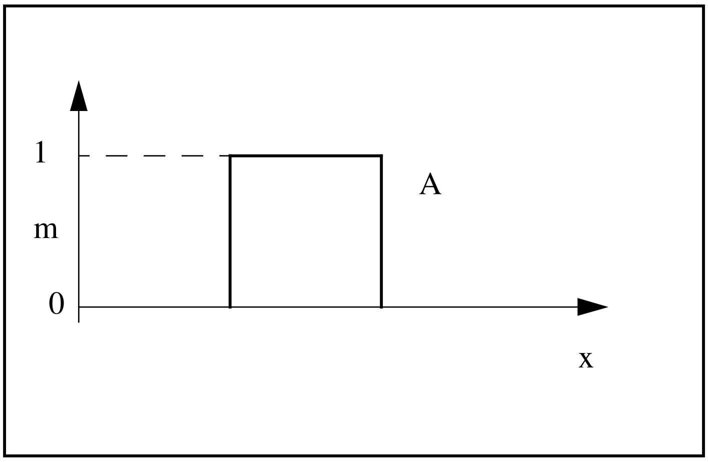


Fig 2-7. Membership function for crisp set A.

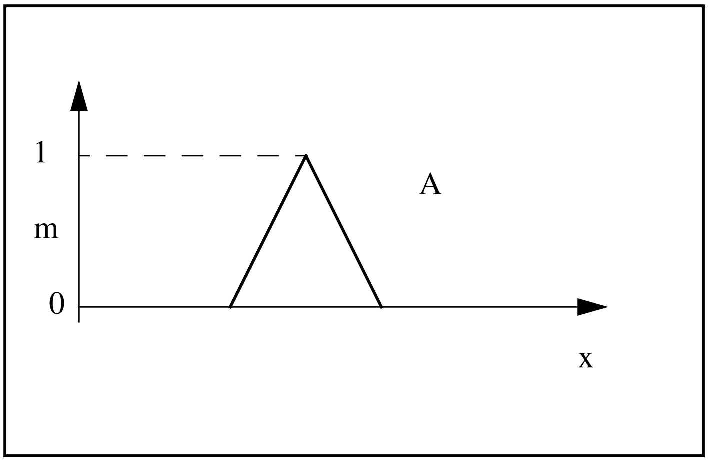


Fig 2-8. Membership function for fuzzy set $\underset{\sim}{A}$

A notation convention for fuzzy sets that is popular in the literature when the univers of discourse, X, is discrete and finite, is given bellow for a fuzzy set $\underset{\sim}{A}$ by:

$$\underset{\sim}{A} = \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \ldots\ldots = \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i}$$

and when the univers , X, is continuous and infinite, the fuzzy set $\underset{\sim}{A}$ is denoted by:

$$\underset{\sim}{A} = \int \frac{\mu_{\underset{\sim}{A}}(x)}{x}$$

In both notation, the horizontal bar is not a quotient, but rather a delimiter. The numerator in each individual expression is the membership value in set $\underset{\sim}{A}$ associated with the element of the univers indicated in the denominator of each expression. In yhe first notation, the summation symbol is not for algebraic summation, but rather is denoting fuzzy union; hence the “+” signs in the first notation are not algebraic “add”, but rather function theoretic union. in the second notation the integral sign is not an algebraic integral, but rather a set union notation for continuous variables.

1.2.2.1 Fuzzy Set Operations

Define three fuzzy sets $\underset{\sim}{A}$ $\underset{\sim}{B}$ $\underset{\sim}{C}$ on the univers X. For a given element x of the universe, the following function theoretic operations for the set-theoretic operations of union, intersection, and complement are defined:

$$\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C} on X$$

Union:

$$\mu_{\underset{\sim}{A} \cup \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \vee \mu_{\underset{\sim}{B}}(x)$$

Intersection:

$$\mu_{\underset{\sim}{A} \cap \underset{\sim}{B}}(x) = \mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(x)$$

Complement:

$$\mu_{\overline{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x)$$

Venn diagrams for these operations, extended to consider fuzzy sets, are shown in Figures.2-9 through 2-11.

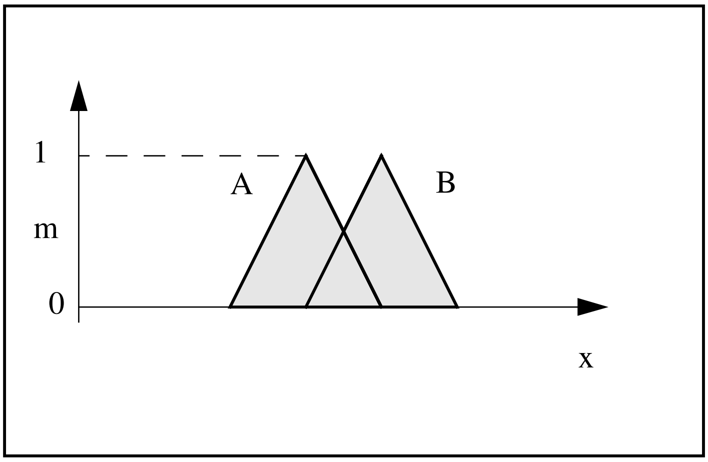


Figure 2-9. Union of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

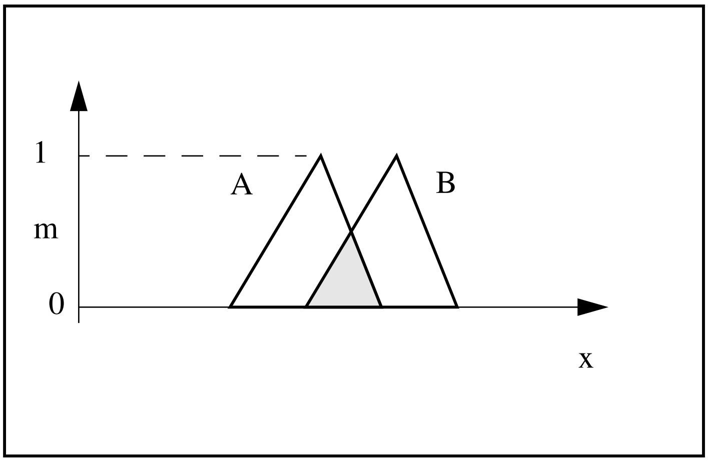


Figure 2-10. Intersection of fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$

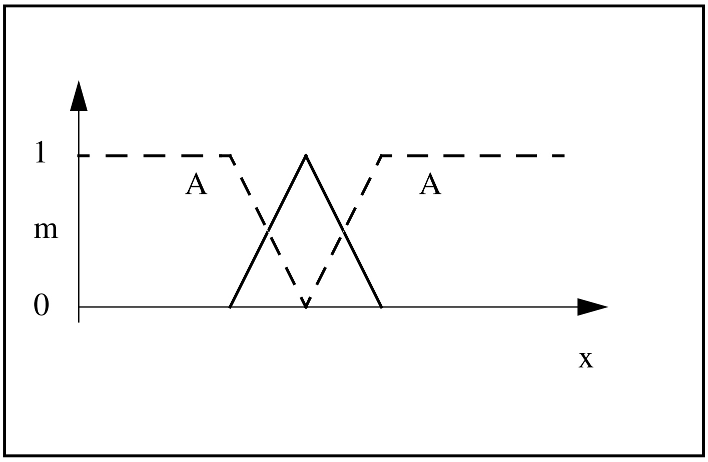


Figure 2-11. Complement of fuzzy sets $\underset{\sim}{A}$.

Any fuzzy set $\underset{\sim}{A}$ defined on a univers X is a subset of that univers. Also by definition, just as with classical set , the membership value of any element x in the null set $\varnothing$ is 0 and the mem-

bership value of any element x in the whole set X is 1. Note that the null set and the whole set are not fuzzy sets in the context (no tilde understrike).
the appropiate notation of that ideas is as follows:

$$\underset{\sim}{A} \subseteq X \rightarrow \mu_{\underset{\sim}{A}}(x) \leq \mu_{x}(x)$$

$$\forall x \in X, \mu_{\varnothing}(x) = 0$$

$$\forall x \in X, \mu_{x}(x) = 1$$

The collection of all fuzzy sets and fuzzy subsets on X is denoted as the fuzzy power set $\underset{\sim}{P}(X)$. It should be obvious to see, based on the fact that all fuzzy sets can overlap, that the cardinality of the fuzzy power set is infinite; that is, Cardinality of

$$P(x) \rightarrow n_{p(x)} = \infty$$

De Morgan's laws for classical sets also hold for fuzzy set, as denoted by the expression below,

$$\overline{(\underset{\sim}{A} \cap \underset{\sim}{B})} = \bar{\underset{\sim}{A}} \cup \bar{\underset{\sim}{B}}$$

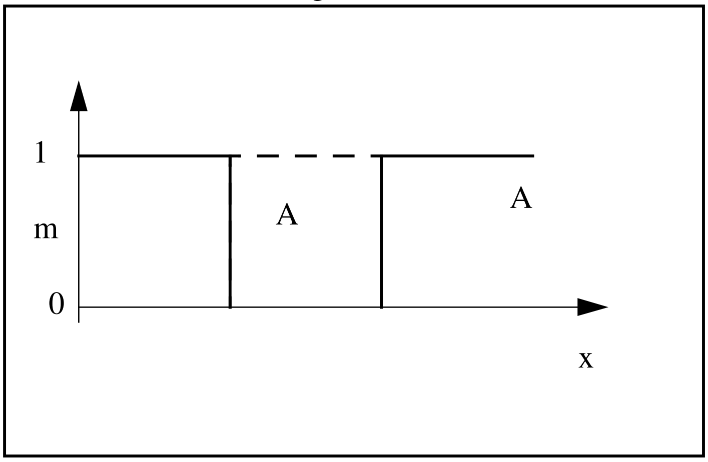
$$\overline{(\underset{\sim}{A} \cup \underset{\sim}{B})} = \bar{\underset{\sim}{A}} \cap \bar{\underset{\sim}{B}}$$

All other operations on classical sets, as enumerated before, also hold for fuzzy sets, exept for the excluded middle laws. These two laws do not hold for fuzzy sets because of the fact that since fuzzy sets can overlap, a set and its complement can also overlap. The exluded middle laws, extended for fuzzy sets, are expressed by,

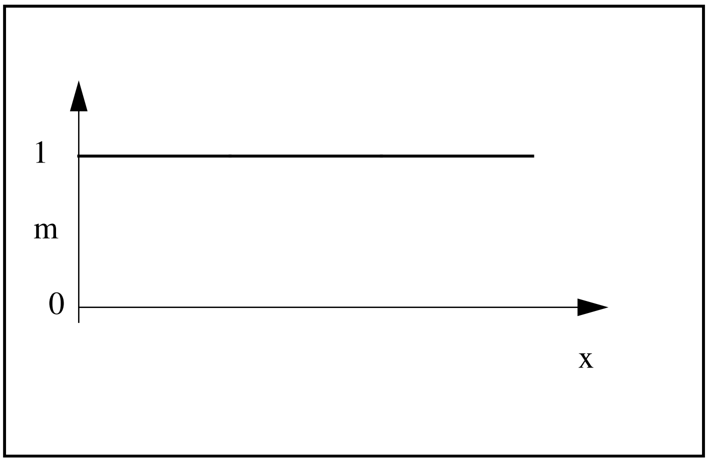
$$\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} \neq \varnothing$$

$$\underset{\sim}{A} \cup \overline{\underset{\sim}{A}} \neq X$$

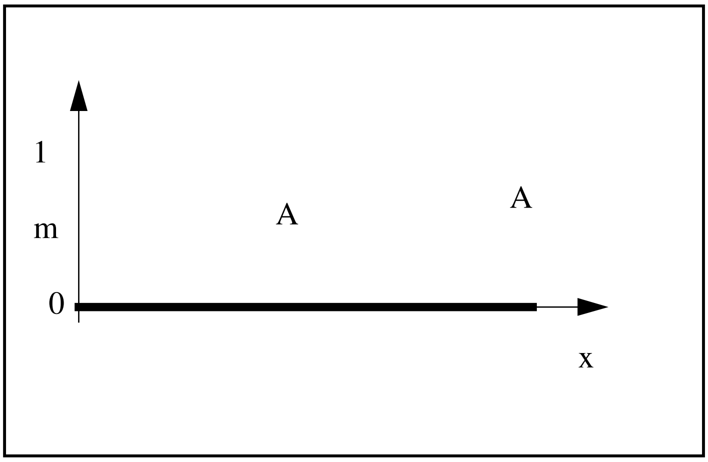
Extended Venn diagrams for these situations, and comparisons to the excluded middle laws for classical (crisp0 sets, are shown below in Figures 2-12 and 2-13.



a) Crisp set A and its complement

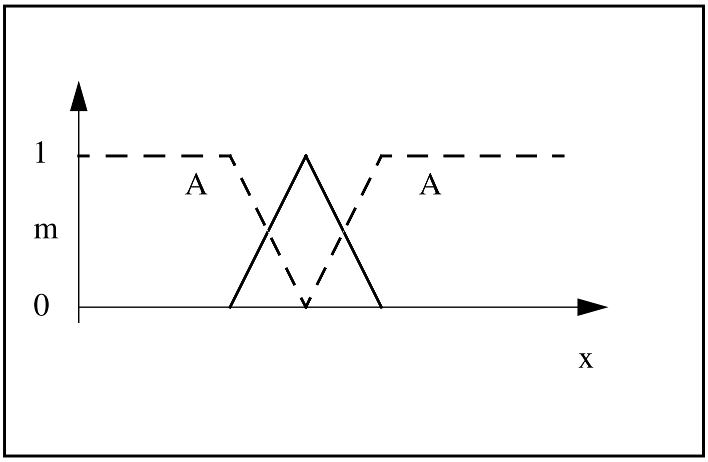


b) Crisp $A \cup \overline{A} = X$.

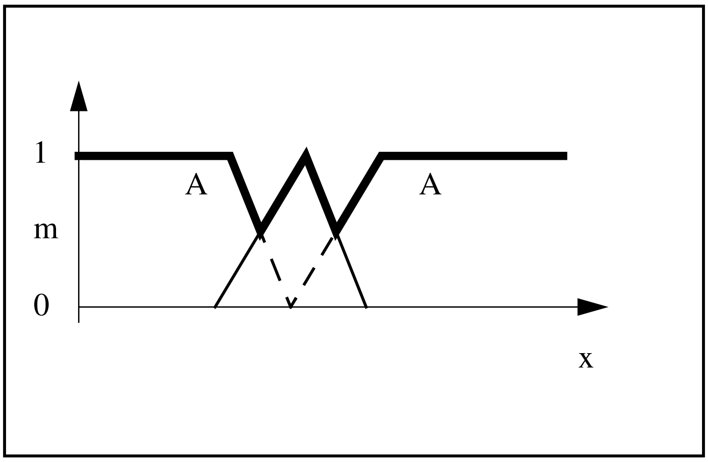


c) Crisp $A \cap \bar{A} = \emptyset$

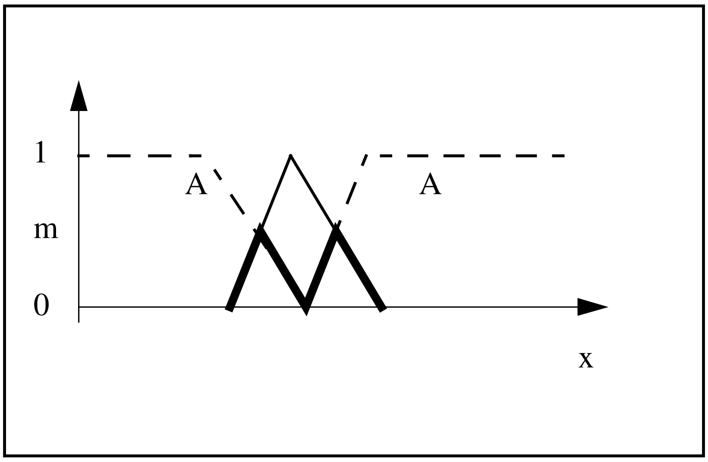
Figure 2-12. Excluded Middle Laws for crisp sets.



a) Fuzzy set A and its complement



b) Fuzzy $\underset{\sim}{A} \cup \underset{\sim}{\bar{A}} = X$.



c) Fuzzy $\underset{\sim}{A} \cap \overline{\underset{\sim}{A}} = \varnothing$

Figure 2-13. Excluded Middle Laws for fuzzy sets.

1.2.2.2 Properties of fuzzy Sets.

Fuzzy sets follow the same properties as do crisp sets. Because of this and because of the membership values of a crisp set are a subset of interval [0,1], classical setscan be thought of as a special case of fuzzy sets. Frequently used properties of fuzzy sets are listed below.

Commutivity:

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

Associativity:

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cup \underset{\sim}{C}$$

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cap \underset{\sim}{C}$$

Distributivity:

$$\underset{\sim}{A} \cup (\underset{\sim}{B} \cap \underset{\sim}{C}) = (\underset{\sim}{A} \cup \underset{\sim}{B}) \cap (\underset{\sim}{A} \cup \underset{\sim}{C})$$

$$\underset{\sim}{A} \cap (\underset{\sim}{B} \cup \underset{\sim}{C}) = (\underset{\sim}{A} \cap \underset{\sim}{B}) \cup (\underset{\sim}{A} \cap \underset{\sim}{C})$$

Idempotency:

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

Identity:

$$\underset{\sim}{A} \cup \varnothing = \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \varnothing = \varnothing$$

$$\underset{\sim}{A} \cap X = \underset{\sim}{A}$$

$$\underset{\sim}{A} \cup X = X$$

Transitivity:

$$\text{If } \underset{\sim}{A} \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{C}, \text{ Then } \underset{\sim}{A} \subseteq \underset{\sim}{C}$$

Involution:

$$\overline{(\overline{\underset{\sim}{A}})} = \underset{\sim}{A}$$

1.2.2.3 Feature of membership function

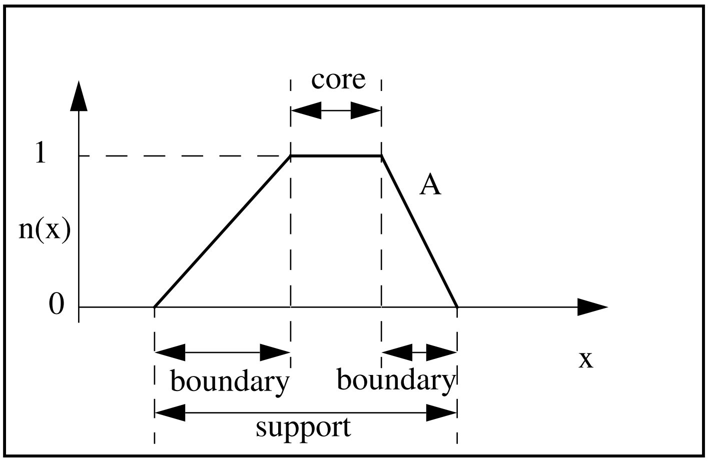


Figure 2-14. Core, support, and boundaries of a fuzzy set.

The core of a membership function for some fuzzy sets $\underset{\sim}{A}$ is defined as that region of the univers that is characterized by complete and full membership in the set $\underset{\sim}{A}$. That is, the core is comprised of those elements of the univers, x, where $\mu_{\underset{\sim}{A}}(x) = 1$.

The support of a membership function for same fuzzyset $\underset{\sim}{A}$ is defined as that region of the univers that is characterized by complete and full membership in the set $\underset{\sim}{A}$. That is, the support is comprised of those elements of the univers, x, where $\mu_{\underset{\sim}{A}}(x) \neq 0$

The boundaries of a membership function for same fuzzy set $\underset{\sim}{A}$ are defined as the region of the univers that contains elements that have a non-zero membership, but not complete complete membership. That is, the boundaries are comprised of those elements of the univers, x, where $0 < \mu_{\underset{\sim}{A}}(x) < 1$. These elements of the univers are those with some “degree” of fuzziness, or only partial membership in the fuzzy set $\underset{\sim}{A}$. Figure 2-14 illustrates the regions in the univers comprising the core, support, and boundaries of a typical fuzzy set.

A normal fuzzy set is one whose membership function has the least one element in the univers, x, whose membership value is unity. For fuzzy sets where one and only one element has a membership equal to one, this element is typically refferred to as the "prototype" of the set. Figures 2-15and 2-16 illustrates typical **normal** and **non-normal** fyzzy sets.

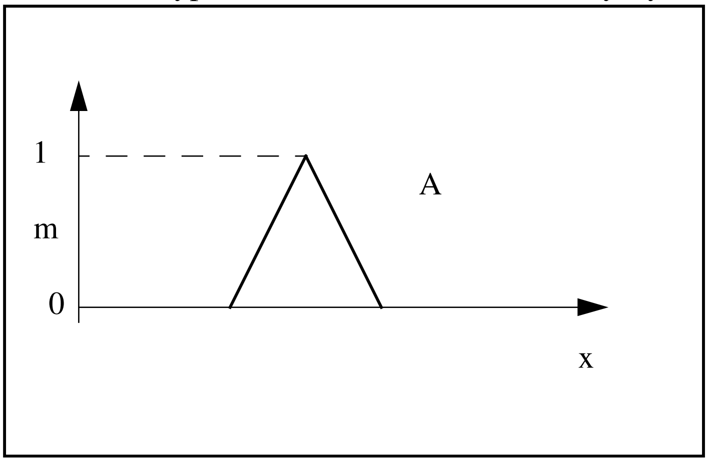


Figure 2-15. A normal fuzzy set.

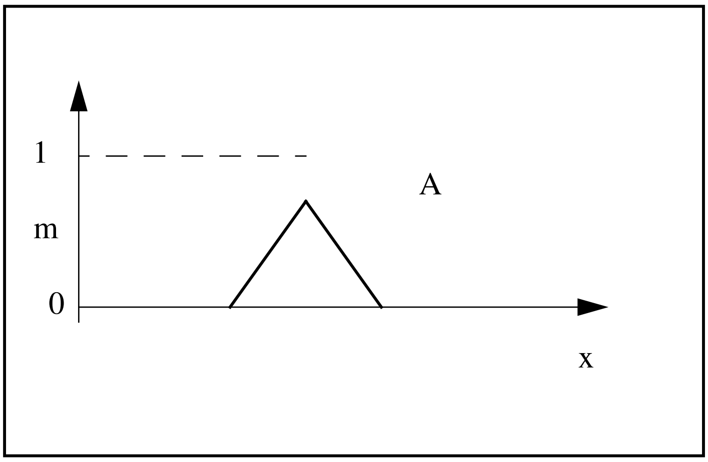


Figure 2-16. A non-normal fuzzy set.

A **convex** fuzzy set is described by a membership function whose membership value are strictly monotonically increasing, or whose membership values are strictly monotonically decresing , or whose membership values are strictly monotonically incresing than strictly monotonically decresing with incresing values of elements in the univers. said anothe way, if for all elements in a continous fuzzy set $\underset{\sim}{A}$ where x<y<z, and where $\mu_{\underset{\sim}{A}}(y) \geq min\,[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)]$ then $\underset{\sim}{A}$is to be said to be a convex fuzzy set.

Figures 2-17 and 2-18 shows a typical **convex** fuzzy set and a typical **non-convex** fuzzy set.

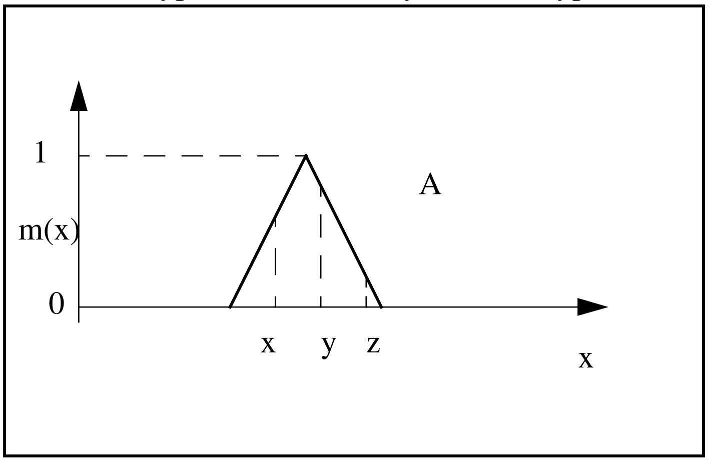


Figure 2-17. Convex Fuzzy Set

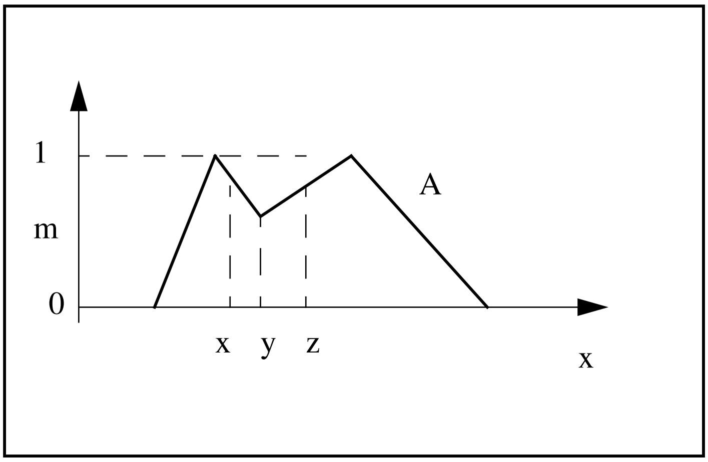


Figure 2-18. Non-Convex Fuzzy Set

It should be noted, that a special property of two convex sets, say $\underset{\sim}{A}$ and $\underset{\sim}{B}$, is tha the intersection of these two convex fuzzy sets are also a convex fuzzy set, as shown in the next figure. For $\underset{\sim}{A}$ and $\underset{\sim}{B}$, which are both convex, $\underset{\sim}{A} \cap \underset{\sim}{B}$ is also convex.

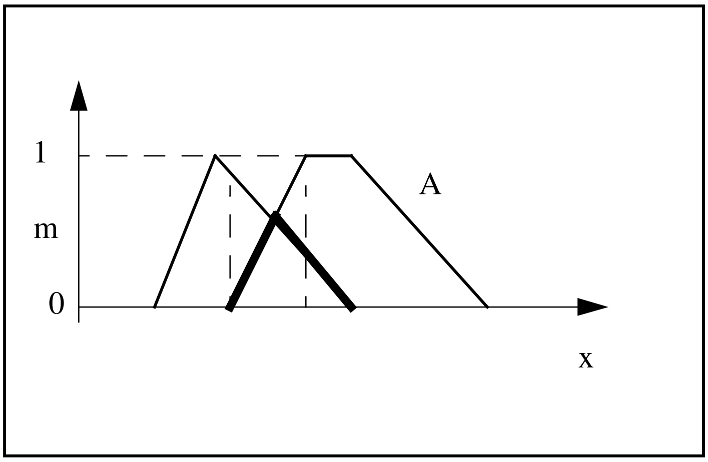


Figure 2-19. the intersection of two convex fuzzy sets.

1.3 The Extension Principle

Extension Principle

Heretofore we have discussed features of fuzzy sets on certain univers of discourse. Suppose there is a mapping between elements, u, of univers U, onto elements, v, of another univers, V, through a function f. An extension principle developed by Zadeh (1975) and later elaborated by Yager 91986) allows us to extend the domain of a function on fuzzy sets. let this mapping be described by,

Let $f{:}u \to v$

Define $\underset{\sim}{A}$ to be a fuzzy set on univers U; that is, $\underset{\sim}{A} \subset U$, and

$$\underset{\sim}{A} = \frac{\mu_1}{u_1} + \frac{\mu_2}{u_2} + \dots \frac{\mu_n}{u_n}$$

Then the extension principle asserts that, for a function f that maps one element in univers U to one element in univers V,

$$f(\underset{\sim}{A}) = f\left(\frac{\mu_1}{u_1} + \frac{\mu_2}{u_2} + \dots \frac{\mu_n}{u_n}\right) = \frac{\mu_1}{f(u_1)} + \frac{\mu_2}{f(u_2)} + \dots \frac{\mu_n}{f(u_n)}$$

This mapping is said to be “one-to-one”

==

Example: Let a fuzzy set $\underset{\sim}{A}$ be defined on the univers U={1,2,3}. We wish to map elements of this fuzzy set to another univers, V, under the function

v=f(u)=2u-1

we see that the elements of V are, V={1,3,5}. So, for example suppose the fuzzy set $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A} = \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3}$$

then the fuzzy membership function for v=f(u)=2u-1 would be

$$f(\underset{\sim}{A}) = \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3}$$

==

For cases where this functional mapping f maps products of elements from the univers, say U_1 and U_2, to another univers v, and we define $\underset{\sim}{A}$ as a fuzzy set in cartesian space $U_1 \times U_2$, then

$$f(A) = \{ \frac{min[\mu_1(i), \mu_2(i)]}{f(i,j)} \underset{\sim}{} \backslash i \in U_1, j \in U_2 \}$$

where μ_1(i) and μ_2(j) are the separable membership projection of μ(i,j) from the Cartesian space $U_1 \times U_2$ when μ(i,j) can not be determined. This projection involves the invocation of a condition known as "non-interaction" between the separate universes, and is analogous to the assumption of independence employed in probability theory which reduces a joint probability density function to the product of its separate marginal density functions. In the fuzzy non-interaction case we use the minimum function as opposed to the product function used in probability theory.

==

Example: Suppose we have the integers 1 to 10 as the elements of two identical but different universes. Let U_1=U_2={1,2,3,.....,10}, then define two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ on univers U_1 and U_2 respectively,

Define $\underset{\sim}{A} = \underset{\sim}{2} =$ *"approximatly 2"*$= \frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3}$ *and*

define $\underset{\sim}{B} = \underset{\sim}{6} =$ *"approximatly 6"*$= \frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7}$,

then the product of ('approximatly 2") ("aproximately 6")should map to a fuzzy number "approximately 12", which is a fuzzy set defined on a univers, say V, of integers, V={5,6,...18,21}, as determined by the extension principle,*

$$\underset{\sim}{2} \times \underset{\sim}{6} = \left(\frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3} \right) \times \left(\frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7} \right)$$

$$= \frac{min\,(0.6, 0.8)}{5} + \frac{min\,(0.6, 1)}{6} + \ldots\ldots + \frac{min\,(0.8, 1)}{18} + \frac{min\,(0.8, 0.7)}{21}$$

$$= \frac{0.6}{5} + \frac{0.6}{6} + \frac{0.6}{3} + \frac{0.8}{10} + \frac{1}{12} + \frac{0.7}{14} + \frac{0.8}{15} + \frac{0.8}{18} + \frac{0.7}{21}$$

==

The complexity of the extension principle increases whwn we consider if more than the combination of the input variables, U_1 and U_2, are mapped to the same variable in the output space, V> in this case we take the maximum membership grades of the combinations mapping to the same output variable, or for the mapping shown below we get,

$$\mu_{\underset{\sim}{A}}(u_1, u_2) = max\left[min\,\{\mu_1(u_1), \mu\ \ (u_2)\}\right], \nu = f(u_1, u_2)$$

==

Example: we have two fyzzy sets $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$*, each defined on its own universe, as follows,* $\underset{\sim}{a} = \frac{0.2}{1} + \frac{1}{2} + \frac{0.7}{3}$ *and* $\underset{\sim}{b} = \frac{0.5}{1} + \frac{1}{2}$

We wish to determine the membership values for the mapping

$$f(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{a} \times \underset{\sim}{b} = \frac{min\,(0.2, 0.5)}{1} + \frac{max\,[min\,(0.2, 1), min\,(0.5, 1)]}{2}$$

$$+ \frac{max\,[min\,(0.7, 0.5), min\,(1, 1)]}{4} + \frac{min\,(0.7, 1)}{8} = \frac{0.2}{1} + \frac{0.5}{2} + \frac{1}{4} + \frac{0.7}{8}$$

In this case, the mapping involves two ways to produce a 2(1x2 and 2x1) and two ways to produce a 4 (4x1 and 2x2).

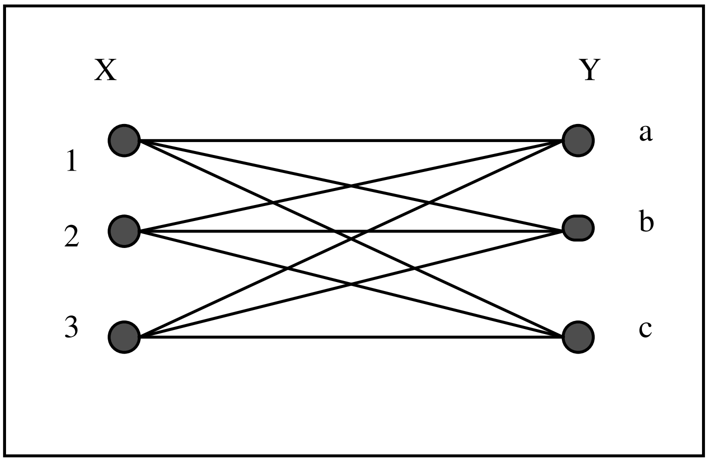
==

1.4 Relation among Classical (crisp) sets

The Cartesian product of two universes X and Y is determined as

$$X \times Y = \{ (x, y) / \mathrm{x} \in X, y \in Y \}$$

which combines $\forall x \in X and \forall y \in Y$ in an ordered pair and forms unconstrained matches between x and y. that is, every element in univers X related completly to every elenment in univers Y. the “strength” of this relationship between ordered pairs of elements in each univers is measured by the caracteristic function, where a value of unity is associated with “complete relationship” and a value of zero is associated with “ no relationship”, i.e., the binary values 1 and 0. An example is given in the Sagittal diagram shown below, and in the matrix expression to follow, where values of unity in the relation matrix, denoted R, correspond to the ordered pairs of mappings in the relation.



$$\begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 2-20. sagittal diagrams (top) and matrix expressions (bottom) of an unconstrained relation

A more general crisp relation ,R, exist when matches between elements in two universe are constrained. again, the caracteristic function is used to assign values of relationship in the mapping of Chatrtesian space XxY to the binary values of (0,1).

$$\chi_R(x, y) = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

==

Example: In many biological models, members of certain species can reproduces only with certain members of another species. Hence, only some elements in two ore more universes are paired. An example is shown below for two 2-member species. $R = \{(1, a), (2, b)\}$

$$R \subset X \times Y \quad R = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

==

1.4.1 Cardinality of Crisp Relations

Suppose n elelments of the univers X are related to m elements of the univers Y. If the cardinality of X is n_X and the cardinality of Y is n_Y, then the cardinality of the relation, R, between these two universes is $n_{X*Y} = n_X * n_Y$. The cardinality of power set describing this relation , P(X*Y) is then $n_{P(X*Y)} = 2^{(nX*nY)}$.

1.4.2 Operation on Crisp Relation

Define R and S as two separate relations on Cartesian universe XxY, and define the null relation and the complete relation as the matrices O and E, respectively. An example of a 4x4 form of the O and E matricies is given bellow:

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The following function-theoretic operations for the two crisp relations (R,S) can now be defined.

Union:

$$R \cup S \rightarrow \chi_{R \cup S}(x, y) = \ max\,[\chi_R(x, y), \chi_S(x, y)]$$

Intersection:

$$R \cap S \rightarrow \chi_{R \cap S}(x, y) = \ min\,[\chi_R(x, y), \chi_S(x, y)]$$

Complement:

$$\bar{R} \rightarrow \chi_{\bar{R}}(x, y) = \ 1 - \chi_{\bar{R}}(x, y)$$

Containment:

$$R \subset S \rightarrow \chi_R(x, y) \ \leq \chi_S(x, y)$$

Identity:

$$(\varnothing \rightarrow OandX \rightarrow E)$$

Further, the operations of commutativity, associativity, distributivity, involution, and idempotency all holds just as they do for classical set operations. Morover, DeMorgan's laws and the Excluded Middle laws also hold for crisp (classical) relations just as they do for crisp (classical0 sets. The null relation, O, and the complete relation, E, are analogous to the null set and the whole set in set-theoretic form.

Now let R be a relation that relates, or maps, elements from univers X to univers Y, and let S be a relation that relates, or maps, element from univers Y to univers Z. Can we find a relation, T, that relates the same elements in univers X That R handles to the same elements in univers Z that S handles? The answer is yes, and we do this with an operation known as composition. So for the Sagittal diagrams in Figure 2-21, we wish to find a relation T that relates the ordered pair (X_1,Z_2); i.e., $(X_1, Z_2) \in T$

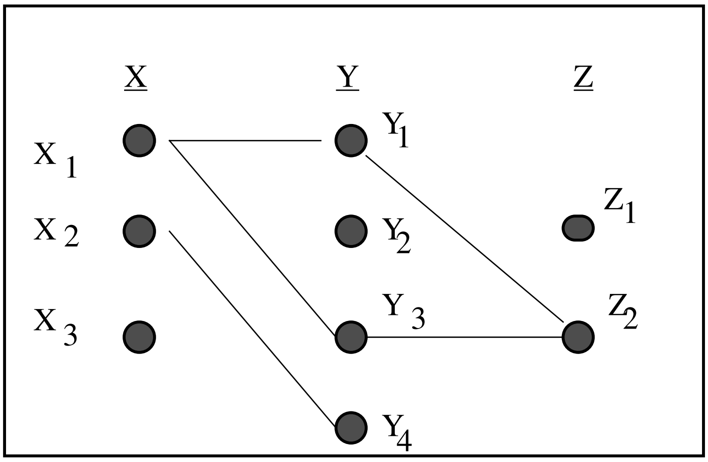


Figure 2-21. sagittal diagram relating elements of three universes.

There are two common forms of the composition operation; one is called the max-min composition and the orther is termed the max-product composition. The max-min compositon is defined by the expression,
T=RoS

$$\chi_T(x,z) = \vee\ (\chi_R(x,y)\ \wedge \chi_S(y,z))\ ;y \in\ Y$$

and the max-product (sometimes called max-dot) composition is defined by the expression,
T=RoS

$$\chi_T(x,z) = \vee\ (\chi_R(x,y)\ \cdot \chi_S(y,z))\ ;y \in\ Y$$

===

Example: The matrix expression for the crisp relations shown in Figure 2-21 can be found using the max-min composition operation. Relation R and S would be expressed as,

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ;S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow T\ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

===

1.5 Relation among fuzzy sets

Fuzzy relations also map elements of the universe, say X, to those of another universe, say Y, through the Cartesian product of the two universes. However, the ‘strength’ of the relation between ordered pairs of the two universes is not measured with the charecteristic function , but rather with a membership function expressing various “degrees’ of the strength of the relation on the unit interval [0,1]. hence, a fuzzy relation $\underset{\sim}{R}$, is a mapping from the Cartesian space XxY to the interval [0,1], where the strength of the mapping is expressed by the membership function of the relation for ordered pairs from the two universes, or $\mu_{\underset{\sim}{R}}(x, y)$.

1.5.1 Cardinality of Fuzzy Relations

Since the cardinality of fuzzy sets on any univers is infinity, then the cardinality of a fuzzy relation between two or more universes is also infinity.

Operations on fuzzy relations

Let $\underset{\sim}{R}$, $\underset{\sim}{S}$, and $\underset{\sim}{T}$ be fuzzy relations on the Cartesian space XxY. Then the following operations apply,

Union:

$$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x, y) = max(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$$

Intersection:

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x, y) = min(\mu_{\underset{\sim}{R}}(x, y), \mu_{\underset{\sim}{S}}(x, y))$$

Complement:

$$\mu_{\overline{\underset{\sim}{R}}}(x, y) = 1 - \mu_{\underset{\sim}{R}}(x, y)$$

Containment:

$$(\underset{\sim}{R} \subset \underset{\sim}{S}) \Rightarrow \mu_{\underset{\sim}{R}}(x, y) \le \mu_{\underset{\sim}{S}}(x, y)$$

Just as for crisp relations, the operation of commutativity, associtivity, distributivity, involution, and idempotency all holds for fuzzy relations. Moreover, DeMorgan's laws hold for fuzzy relations just as they do for crisp relations, and the null relation,O, and the complete relation, E, are analogous to the null set and the whole set in set-theoretic form, respectively. The operations that do not hold for fuzzy relations, as is the case for fuzzy sets in general, are Excluded Middle laws. Since a fuzzy relation $\underset{\sim}{R}$ is also fuzzy set, there is overlap between a relation and its complement, and $\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq E$, $\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq O$

As seen, the Excluded Middle laws for relations do not result in the null relation, O, or the complete relation, E.

Because fuzzy relations in general are fuzzy sets, we can define the Cartesian productbetween fuzzy sets. Let $\underset{\sim}{A}$ be a fuzzy set on universe X and $\underset{\sim}{B}$ a fuzzy set in universe Y.; then the carte-

sian product between fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$ will results in a fuzzy relation $\underset{\sim}{R}$, or

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y$$

with membership function,

$$\mu_{\underset{\sim}{R}}(x, y) = \mu_{\underset{\sim}{A} \underset{\sim}{\times} \underset{\sim}{B}}(x, y) = min(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y))$$

==

Example: Suppose we have two fuzzy sets on a universe, $\underset{\sim}{A}$ and $\underset{\sim}{B}$, and we want to find the fuzzy Cartesian product between them. Let,

$$\underset{\sim}{A} = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} and \ \underset{\sim}{B} = \frac{0.3}{y_1} + \frac{0.9}{y_2}$$

The Cartesian product is,

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix}$$

==

fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose $\underset{\sim}{R}$ is a fuzzy relation on the Cartesian space XxY, $\underset{\sim}{S}$ is a fuzzy relation on YxZ, and $\underset{\sim}{T}$ is a fuzzy relation on XxZ; then the fuzzy max-min composition is defined as: Let $\underset{\sim}{T} = \underset{\sim}{R} \bullet \underset{\sim}{S}$

$$\mu_{\underset{\sim}{T}}(x, z) = \vee (\mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{S}}(y, z)) ; y \in Y$$

and the fuzzy max-product composition is defined as,

$$\mu_{\underset{\sim}{T}}(x, z) = \vee (\mu_{\underset{\sim}{R}}(x, y) \cdot \mu_{\underset{\sim}{S}}(y, z)) ; y \in Y$$

==

Example: Let us extend the information contained in the Sagittal diagram shown in Figure 2-21 to include fuzzy relationship between the universes X-Y, and Y-Z. consider the following *fuzzy relations,*

$$\underset{\sim}{R} = \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \text{ and } \underset{\sim}{S} = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}$$

Then the resulting relation, $\underset{\sim}{T}$, which relates elements of universe X to elements of universe Z, can be found by max-min composition to be,

$$\underset{\sim}{T} = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix}$$

==

It should be pointed out that neither crisp not fuzzy compositions have inverses in general; that is

$$\underset{\sim}{R} \bullet \underset{\sim}{S} \neq \underset{\sim}{S} \bullet \underset{\sim}{R}$$

This results in general for any matrix operation, fuzzy or otherwise, which must satisfy consistency between the cardinal counts of elements in respective universes. Even for the case of square matrices, the composition inverse is not guaranteed.

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