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Chapter 1 Theoretical Overview

Introduction

This chapter gives an introduction to the fundamental notions and concepts of fuzzy sets. Fuzzy sets were establisheed by L. A. Zadeh in 1965 in his seminal papers [1, 2]. Nowadays, they are equipped with their own mathematical foundations, rooting from set-theoretic basis and many-valued logic. Their achievements have alredy enriched the classic two-valued calculus with a deep and novel perspective.

To understand the reasons for this extensive develpoment of fuzzy sets, there are two main aspects worthy of being mentioned. Firstly, the notion of fuzzy set, as a tool for modelling intermediate grades of belonging that occur in any concept, is very attractive, especially from an applicational point of view. Secondly, a variety of tools incorporated in the framework of fuzzy sets enables to find a suitable concept to cope with reality.

It is not our aim to give an exhaustive presentation of the entire background of fuzzy sets, but to provide the reader with basic and selected knowledge in this area, with great emphasis on ideas having strong applicational links. We do not intend to move towards the abstract fields of development of fuzzy sets with highly advanced mathematical formalism (for examle, topological fuzzy spaces, fuzzy algebra or fuzzy category theory).

To introduce the idea of fuzzy set, let us remind ourselves of two-valued logic, which forms a cornerstone of any mathematical tool used. A fundamental point arising from this logic is that it imposes a dichotomy of any mathematical model. In other words, taking any object, we are forced to assign it to one of two prespecified categories (for example, good-bad, black-white, normal-abnormal, odd-even, etc.

Sometimes it happens that this process of classification (discrimination) may easily performed, since the categories we are working with are precise and well-defined. For instance, with two categories of natural numbers, odd and even, we can classify any natural number as belonging to exatly one class.

Nevertheless, in many engineering tasks, we are faced with classes that are ill-defined. Consider, for instance, such categories as *tall* man, *high* speed, *significant* error, etc. All of these convey a useful semantic meaning that is obvious for a certain community. However, a borderline between the belonging or not of a given object to such a class is not evident. Here, it is obvious that two-valued logic, used in describing these classes of situations, might be not well-suited.

An historical example appeared in one of the works Borel [3], who discussed an ancient Greek sophism of the pile of seeds,

> "... one seed does not constitute a pile nor two nor three ... from the other side everybody will agree that 100 million seeds constitute a pile. What therefore is the appropriate limit? Can we say that 325 647 seeds don't constitute a pile but 325 648 do?"

Also, even in mathenatics we can meet some fuzzy notations. Examples of such expressions which are well known to control engineers: *sparse* matrix, a linear approximation of a function in a *small* neighbourhood of a point x_0, or an *ill-conditioned* matrix. Here, we accept the notions as conveying useful information, not treating the as an evident defect of everyday language. Nevertheless, we should state that these notions are strongly context-dependent: they depend, for example, on the type of computer used (i.e. keeping track of the idea of an ill-defined matrix, for instance). Notice, however, that the fuzzyness presented in the above facts has totally different character in comparison with randomness.

The above-sketched examples prompt us to become acquanted with some of the different fields of fuzzy logic, such as fuzzy measure, measure of fuzzyness, fuzzy logic and fuzzy reasoning.

1.1. Fuzzy Sets

1.1.1. Fuzzy Sets and Terminology

Let U be a collection of objects denoted generically by $\{u\}$, which could be discrete or continuous. U is called the universe of discourse and u represents the generic element of U.

Definition 1-1 (fuzzy set): A fuzzy set F in a universe of discourse U is characterized by a membership function μ_F which takes values in the interval [0, 1] namely μ_F: $U \rightarrow [0, 1]$. A fuzzy set may be viewed as a generalization of the concept of an ordinary set whose membership function only takes two values {0, 1}. Thus fuzzy set F in U may be represented as a set of ordered pairs of a generic element u and its grade of membership function: $F\{(u, \mu_F(u))|\ u \in U\}$ (Fig. 1.1). When U is continuous, a fuzzy set F can be written concisely as $F = \int_u \mu_F(u)/u$. When U is discrete, a fuzzy set F is represented as

$$F = \sum_{i=1}^{n} \mu_F(u_i)/u_i,$$

where the / (slash) is employed to link the elements of the support with their grades of membersip in F, and the sign of summazion indicates, rather than any sort of algebric summation, that the listed pairs of elements and membersip grades collectively form the definition of the sat F.

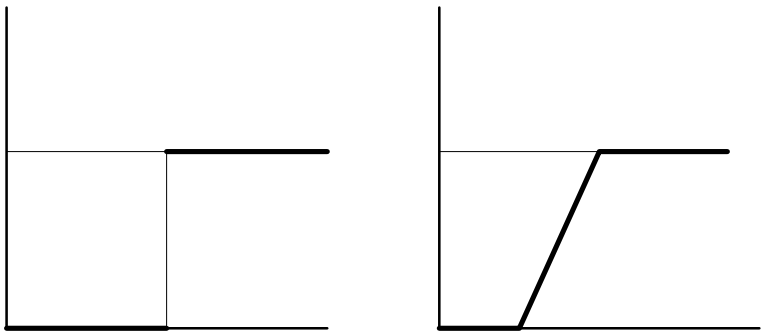

Figure 1-1. The binary (a) and the multilevel (b) membership function

Definition 1-2 (support, crossover point, and fuzzy singleton): The support of a fuzzy set is the crisp set (normal set, whose membership function is two valued) of all points u in U such that $\mu_F(u) > 0$, that is, an empty fuzzy set has an empty support. In particular, the element u in U at which $\mu_F(u) = 0.5$, is called the crossover point and a fuzzy set whose support is a single point in U with $\mu_F(u) = 1.0$ is referred to as fuzzy singleton (s). There is no singleton of the figure 1.1.b since the support of the $\mu_F(u) = 1.0$ point is not a single point, but an interval.

Definition 1-3 (normalized fuzzy set): A fuzzy set is called normalized when at least one of its elements attains the maximum possible membership grade. If membership grades range in the closed

interval between 0 and 1, for instance, then at least one element must have a membership grade of 1 for the fuzzy set to be considered normalized.

Definition 1-4 (α-cut): An α-cut of fuzzy set F is a crisp set F_α that contains all the elements of the universal set of U that have a membership grade in F greater than or equal to the specified value of α. This definition can be written as $F_\alpha = \{u \in U | \mu_F(u) \geq \alpha\}$.

Definition 1-5 (level set): The set of all levels $\alpha \in [0, 1]$ that represent distinct α-cuts of a given fuzzy set F is called a level set of F. Formally,

$$\Lambda_A = \{\alpha | \mu_A(u) = \alpha \text{ for some } u \in U\}$$

where Λ_A denotes the level set of fuzzy set A defined on U.

Definition 1-6 (convex fuzzy set): A fuzzy set is convex if and only if each of its α-cuts is a convex set. Equivalently we may say that a fuzzy set F is convex if and only if

$$\mu_F(\lambda \mathbf{r} + (1 - \lambda)\mathbf{s}) \geq \min [\mu_F(\mathbf{r}), \mu_F(\mathbf{s})],$$

for all $\mathbf{r}, \mathbf{s} \in \mathbf{R}^n$ and all $\lambda \in [0,1]$. The Figure 1.3 illustrates a convex and a nonconvex fuzzy set on **R**. Note, that the definition of convexity for fuzzy sets does not necessarely mean that the membership function of a convex fuzzy set is also a convex function.

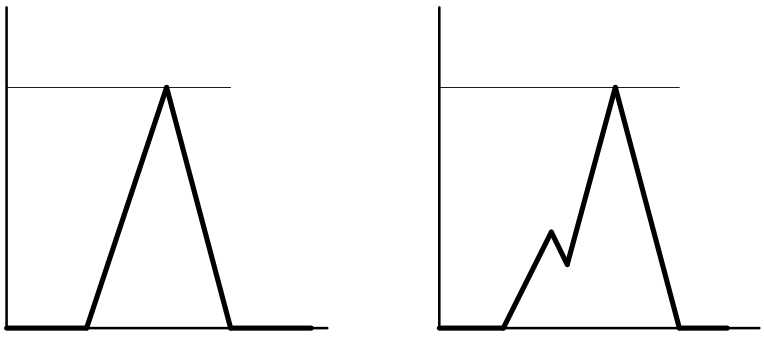

Fig. 1.3. A convex (a) and a nonconvex (b) fuzzy set

Definition 1-7 (fuzzy number): A convex and normalized fuzzy set whose membership function is piecewise continuous is called fuzzy number. Thus, a fuzzy number can be thought of as containing the real numbers within some interval to varying degrees. For example, the membership function of the Figure 1.3.a can be viewed as a representation of fuzzy number 4.

Definition 1-8 (scalar cardinality): The scalar cardinality of a fuzzy set F defined on a finite universal set U is the summation of the membership grades of all the elements of U in F. Thus,

$$|A| = \sum_{u \in U} \mu_A(u).$$

Other forms of cardinality have been proposed for fuzzy sets. One of these, which is called fuzzy cardinality, is defined as a fuzzy number rather than as a real number, as it is the case for scalar cardinality.When fuzzy set A has a finite support, its fuzzy cardinality $|A|$ is a fuzzy set (fuzzy number) defined on **N** whose membership function is defined by

$$\mu_{|A|}(|A_\alpha|) = \alpha,$$

for all α in the level set of $F(\alpha \in \Lambda_F)$.

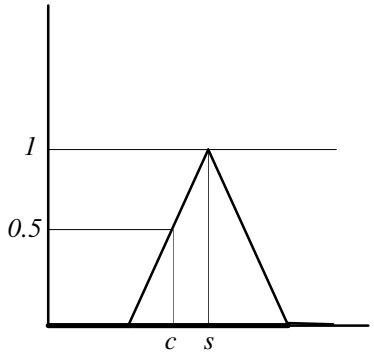

Figure 1-2. The singleton (s) and the crossover point (c)

Definition 1-9 (fuzzy subset): If the membership grade of each element of the universal set *U* in fuzzy set *A* is less than or equal to its membership grade in fuzzy set *B*, then *A* is called a subset of set *B*, that is, if

$$\mu_A(u) \leq \mu_B(u)$$

for every $u \in U$, then

$$A \subseteq B.$$

Definition 1-10 (equal fuzzy sets): Fuzzy sets *A* and *B* are called equal if $\mu_A(u) = \mu_B(u)$ for every element $u \in U$. This denoted by

$$A = B.$$

Definition 1-11 (proper subset of fuzzy set): Fuzzy set *A* is called proper subset of fuzzy set *B* when *A* is a subset of *B* and the two sets are not equal; that is, $\mu_A(u) \leq \mu_B(u)$ for every $u \in U$ and $\mu_A(u) < \mu_B(u)$ for at least one $u \in U$. We can denote this by writing

$$A \subset B \quad \text{if and only if} \quad A \subseteq B \quad \textit{and } A \neq B.$$

1.2. Set Theoretic Operations

1.2.1. Basic Definitions

Let A and B be two fuzzy sets in U with membership functions μ_A and μ_B, respectively. The set theoretic operations of union, intersection and complement for fuzzy sets are defined via their membership functions. More specifically, see the following.

Definition 2-1 (Zadeh-union): The membership function $\mu_{A\cup B}$ of the union $A \cup B$ is pointwise defined for all $u \in U$ by

$$\mu_{A\cup B} = \max\{\mu_A(u), \mu_B(u)\}.$$

Definition 2-2 (Zadeh-intersection): The membership function $\mu_{A\cap B}$ of the intersection $A\cap B$ is pointwise defined for all $u \in U$ by

$$\mu_{A\cap B} = \min\{\mu_A(u), \mu_B(u)\}.$$

It has to be mentioned that min and max are not the only operators that could have been chosen to model the intersection or union of fuzzy sets respectively. The question arises, why those and no others? The problem can be addressed for example axiomatically [4]. It can be argued from a logical point of view, interpreting the intesection as "logical and", the union as "logical or", and the fuzzy set U as the statement "The element u belongs to set U" can be accepted as more or less true. It is very instructiv to follow an example for axiomatic justification of specific mathematical models.

Consider two statements, S and T, for which the truth values are μ_S and μ_T, respectively,

$$\mu_S, \mu_T \in [0, 1].$$

The truth value of the "and" and "or" combination of these statement, μ(S and T) and μ(S or T), both from the interval [0, 1] are interpreted as the values of the membership functions of the intersection and union, respectively, of S and T. We are now looking for two real-valued functions f and g such that

$$\mu_{S \text{ and } T} = f(\mu_S, \mu_T)$$
$$\mu_{S \text{ or } T} = g(\mu_S, \mu_T).$$

The following restrictions are reasonably imposed on f and g:

i. f and g are nondecreasing and continuous in μ_S and μ_T.

ii. f and g are symmetric, that is,

$$f(\mu_S, \mu_T) = f(\mu_T, \mu_S)$$
$$g(\mu_S, \mu_T) = g(\mu_T, \mu_S).$$

iii. $f(\mu_S, \mu_S)$ and $g(\mu_S, \mu_S)$ are strictly increasing in μ_S.

iv. $f(\mu_S, \mu_T) \le \min(\mu_S, \mu_T)$ and $g(\mu_S, \mu_T) \le \max(\mu_S, \mu_T)$. That implies that accepting the truth of the statements "S *and* T" requires more, and accepting the truth of the statements "S *or* T" less than accepting S or T alone as true.

v. $f(1, 1) = 1$ and $g(0, 0) = 0$.

vi. Logically equivalent statements must have equal truth values and fuzzy sets with the same contents must have the same membership function, that is, S_1 *and* (S_2 *or* S_3) is equivalent to (S_1 *and* S_2) *or* (S_2 *and* S_3) and therefore must be equally true.

Let us now formalize the above assumptions as follows: Using the symbols $\wedge$ for "*and*" (= intersection) and $\vee$ for "*or*" (= union), this amounts to the following 7 restrictions, to be imposed on the two commutative (see (ii)) and associative (see (vi)) binary compositions $\wedge$ and $\vee$ on the closed interval [0, 1] which are mutually distributive (see (vi)) with respect to one another.

1. $\mu_S \wedge \mu_T = (\mu_T \wedge \mu_S)$
   $\mu_S \wedge \mu_T = (\mu_T \wedge \mu_S)$

2. $(\mu_S \wedge \mu_T) \wedge \mu_W = \mu_S \wedge (\mu_T \wedge \mu_W)$
   $(\mu_S \vee \mu_T) \vee \mu_W = \mu_S \vee (\mu_T \vee \mu_W)$

3. $\mu_S \wedge (\mu_T \vee \mu_W) = (\mu_S \wedge \mu_T) \vee (\mu_T \wedge \mu_W)$
   $\mu_S \vee (\mu_T \wedge \mu_W) = (\mu_S \vee \mu_T) \wedge (\mu_T \vee \mu_W)$

4. $\mu_S \wedge \mu_T$ and $\mu_S \vee \mu_T$ are continuous and nondecreasing in each component.

5. $\mu_S \wedge \mu_T$ and $\mu_S \vee \mu_T$ are strictly increasing in μ_S (see(iii)).

6. $\mu_S \wedge \mu_T \leq \min(\mu_S, \mu_T)$
   $\mu_S \vee \mu_T \geq \max(\mu_S, \mu_T)$

7. $1 \wedge 1 = 1$
   $0 \vee 0 = 0$ (see(v)).

It can be proven that,

$$\mu_{S \wedge T} = \min(\mu_S, \mu_T) \quad \text{and} \quad \mu_{S \vee T} = \max(\mu_S, \mu_T)$$

For the complement it would be reasonable to assume that if statement "S" is true, its complement "*non*S" is false, or if $\mu_S = 1$ then $\mu_{nonS} = 0$ and vica versa. The function *h* (as complement in analogy to *f* and *g* for intersection and union) should also be continuous and monotonicly decreasing and we would like the complement of the complement to be the original statement. (in order to be in line with traditional logic and set theory). These requirements, however, are not enough to determine uniquely the mathematical form of the complement. It is supposed, in addition, that $\mu_S(1/2) = 1/2$. Other assumptions are certainly possible and plausible.

Definition 2-3 (cartesian product): If $A_1,...,A_n$ are fuzzy sets in $U_1,...,U_n$, respectively, the Cartesian product of $A_1,...,A_n$ is a fuzzy set in the product space $U_1 \times ... \times U_n$ with the membership function

$$\mu_{A_1 \times ... \times A_n}(u_1, u_2, \ldots u_n) = \min \mu_{A_1}(u_1), \ldots \mu_{A_n}(u_n)$$

or

$$\mu_{A_1 \times ... \times A_n}(u_1, u_2, \ldots u_n) = \mu_{A_1}(u_1) \cdot \mu_{A_2}(u_2) \cdot \ldots \cdot \mu_{A_n}(u_n).$$

Definition 2-4 (the mth power): The mth power af a fuzzy set A is a fuzzy set with the membership function

$$\mu_{R^m}(u) = \left[\mu_A(u)\right]^m, \quad u \in U.$$

Definition 2-5 (the algebric sum): The algebric sum $C = A+B$, where A, B, and C are fuzzy sets in U is defined as

$$C = (u, \mu_{A+B}(u) \mid u \in U)$$

where

$$\mu_{A+B}(u) = \mu_A(u) + \mu_B(u) - \mu_A(u) \cdot \mu_B(u)$$

Definition 2-6 (fuzzy complement): The membership function of the complement of a fuzzy set A, $\mu_{\bar{A}}(u)$ is defined by

$$\mu_{\bar{A}}(u) = 1 - \mu_A(u), \quad u \in U.$$

Definition 2-7 (the bounded sum or "bold-intersection"): The bounded sum C = A⊕B is defined as

$$C = (u, \mu_{A \oplus B}(u) \mid u \in U)$$

where

$$\mu_{A \oplus B}(u) = \min\left[1, \mu_A(u) + \mu_B(u)\right].$$

Definition 2-8 (the bounded difference or "bold-union"): The bounded difference C=A−B is defined as

$$C = (u, \mu_{A-B}(u) \mid u \in U)$$

where

$$\mu_{A-B}(u) = \max\left[0, \mu_A(u) + \mu_B(u) - 1\right].$$

Definition 2-9 (the algebric product): The algebric product of two fuzzy sets C=A·B is defined as

$$C = (u, \mu_A(u) \cdot \mu_B(u) \mid u \in U).$$

1.2.2. *t*-norms and *s*-norms for fuzzy sets

For the *intersection of fuzzy sets* Zadeh suggested the min-operator and the algebric product A · B. The "bold intersection" was modeled by the "bounded sum" as defined above. The min, product, and bounded-sum operators belong to the so-called *triangular* or *t-norms.*[1] Operators belonging to this class of *t*-norms are, in particular, associative (see definition 17.4) and therefore it is possible to compute the membership values for the intersection of more than two fuzzy sets by recursively applying a *t*-nom operator [5].These operators belong to the class of *nonparametrized* operators.

definition 2-10 (t-norms): *t*-norms are two-valued functions from [0, 1] × [0, 1] which satisfy the following conditions:

t1. $t(0,\ 0) = t(\mu_A(u),\ 1) = t(1,\ \mu_A(u)) = \mu_A(u) \qquad u \in U.$

t2. $t(\mu_A(u),\ \mu_B(u)) \le t(\mu_C(u),\ \mu_D(u))$

if $\mu_A(u) \le \mu_C(u)$ and $\mu_B(u) \le \mu_D(u).$ (monotonicity)

t3. $t(\mu_A(u),\ \mu_B(u)) = t(\mu_B(u),\ \mu_A(u)).$ (commutativity)

t4. $t(\mu_A(u),\ t(\mu_B(u),\ \mu_C(u))) = t(t(\mu_A(u),\ \mu_B(u)),\ \mu_C(u))$ (associativity)

The functions *t* define the above mentioned general class of intersection operators for fuzzy sets.

Corresponding to the class of *intersection* operators, a general class of aggregation operators for the *union* of fuzzy sets called triangular conorms or t-norms (sometimes referred to as s-norms) is defined analoguously [6, 7]. The max-operator, algebric sum and bounded sum considered above belong to this class

definition 2-11 (t-conorms or s-norms): *t*-conorms or s-norms are associative, commutative, and monotonic two-placed functions s, which map from [0,1] × [0,1] into [0,1]. These properties are formulated with the following conditions:

s1. $s(1,\ 1) = s(\mu_A(u),\ 0) = s(0,\ \mu_A(u)) = \mu_A(u) \qquad u \in U.$

s2. $s(\mu_A(u),\ \mu_B(u)) \le s(\mu_C(u),\ \mu_D(u)).$

if $\mu_A(u) \le \mu_C(u)$ and $\mu_B(u) \le \mu_D(u).$ (monotonicity)

s3. $s(\mu_A(u),\ \mu_B(u)) = s(\mu_B(u),\ \mu_A(u)).$ (commutativity)

s4. $s(\mu_A(u),\ s(\mu_B(u),\ \mu_C(u))) = s(s(\mu_A(u),\ \mu_B(u)),\ \mu_C(u))$ (associativity)

The functions *s* define the above mentioned general class of aggregation operators for fuzzy sets.

t-norms and *t*-conorms are related in sense of *logical duality*, so any *t*-conorm *s* can be generated from *t*-norm *t* through the next transformation.

$$t(\mu_A(u),\ \mu_B(u)) = 1 - s(1-\mu_A(u),\ 1-\mu_B(u)).$$

More general, it is can be showed, that for suitable negation operators like the complement operator for fuzzy sets–see *definition 2-6*–pairs of *t*-norms *t* and *t*-conorms *s* satisfy the following generalization of DeMorgans law [6] (*n* is negation operator)

$$s(\mu_A(u),\ \mu_B(u)) = n(t(n(\mu_A(u)),\ 1-n(\mu_B(u)))) \text{ and}$$

$$t(\mu_A(u),\ \mu_B(u)) = n(s(n(\mu_A(u)),\ 1-n(\mu_B(u)))),\ u \in U$$

Typical dual pairs of nonparametrized *t*-norms and *t*-conorms are compiled below:

$t_w(\mu_A(u),\ \mu_B(u)) = \min(\mu_A(u),\ \mu_B(u))$ if $\max(\mu_A(u),\ \mu_B(u)) = 1,\ 0$ otherwise drastic product

$s_w(\mu_A(u),\ \mu_B(u)) = \max(\mu_A(u),\ \mu_B(u))$ if $\min(\mu_A(u),\ \mu_B(u)) = 1,\ 1$ otherwise drastic sum

$t_1(\mu_A(u),\ \mu_B(u)) = \max(0,\ \mu_A(u) + \mu_B(u) - 1)$ bounded difference

$s_1(\mu_A(u),\ \mu_B(u)) = \min(1,\ \mu_A(u) + \mu_B(u))$ bonded sum

$$t_{1.5}(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)\cdot\mu_B(u)}{2-\left[\mu_A(u)+\mu_B(u)-\mu_A(u)\cdot\mu_B(u)\right]}$$ Einstein product

$$s_{1.5}(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)+\mu_B(u)}{1+\mu_A(u)\cdot\mu_B(u)}$$ Einstein sum

$t_2(\mu_A(u),\ \mu_B(u)) = \mu_A(u)\cdot\mu_B(u)$ algebric product

$s_2(\mu_A(u),\ \mu_B(u)) = \mu_A(u)+\mu_B(u)+\mu_A(u)\cdot\mu_B(u)$ algebric sum

$$t_{2.5}(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)\cdot\mu_B(u)}{\mu_A(u)+\mu_B(u)-\mu_A(u)\cdot\mu_B(u)}$$ Hamacher product

$$s_{2.5}(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)+\mu_B(u)-2\mu_A(u)\cdot\mu_B(u)}{1-\mu_A(u)\cdot\mu_B(u)}$$ Hamacher sum

$t_3(\mu_A(u),\ \mu_B(u)) = \min(\mu_A(u),\ \mu_B(u))$ minimum

$s_3(\mu_A(u),\ \mu_B(u)) = \max(\mu_A(u),\ \mu_B(u))$ maximum

The operators above can be ordered as follows:

$$\begin{aligned} t_w \le t_1 \le t_{1.5} \le t_2 \le t_{2.5} \le t_3 \\ s_w \le s_1 \le s_{1.5} \le s_2 \le s_{2.5} \le s_3 \end{aligned}.$$

We notice, that this order implies that for any fuzzy sets A and B in U with membership values between 0 and 1 any intersection operator that is *t*-norm is bounded by the min-operator and the operator t_w.A *t*-conorm is bounded by the max-operator and the operator s_w, respectively:

$$\begin{aligned} t_w(\mu_A(u),\ \mu_B(u)) \le t(\mu_A(u),\ \mu_B(u)) \le \min(\mu_A(u),\ \mu_B(u)) \\ \max(\mu_A(u),\ \mu_B(u)) \le s(\mu_A(u),\ \mu_B(u)) \le s_w(\mu_A(u),\ \mu_B(u)) \end{aligned}, \qquad u \in U$$

1.2.3. Some parametrized operators

It may be desirable to extend the range of the previouisly described operators in order to adapt them to the context in which they are used. To this end different authors suggested the parametrized families of *t*-norms and *t*-conorms, often maintaning the associativity property.

For illustration porposes we review some interesting parametrized operators. Some of these operators and their equivalence to the logical "and" and "or" respectively has been justified axiomatically. We shall sketch the axioms on which for example the Hamacher-operator rests in order to give the reader the opportunity to compare the axiomatic system was detailed in the previous chapter on one hand with that of the Hamacher-operator on the other.

definition 2-12 (Hamacher-intersection-operator): The intersection of two fuzzy sets A and B is defined as

$$A \cap B = (u,\ \mu_{A\cap B}(u) \mid u \in U)$$

where

$$\mu_{A\cap B}(u) = \frac{\mu_A(u)\cdot\mu_B(u)}{\gamma + (1-\gamma)(\mu_A(u)+\mu_B(u)-\mu_A(u)\cdot\mu_B(u))}\ ,\ \gamma \geq 0$$

Hamacher's basic axioms are as follows:

H1. The operator $\wedge$ is associative, that is $A \wedge (B \wedge C) = (A \wedge B) \wedge C)$.

H2. The operator $\wedge$ is continuous.

H3. The operator $\wedge$ is injective in each argument, that is,

$$A \wedge B = A \wedge C) \Rightarrow B = C$$

$$A \wedge B = C \wedge B) \Rightarrow A = C$$

H3. $\mu_A(u)=1 \Rightarrow \mu_{A\wedge B}(u)=1$

In the above case a function f: $R \rightarrow [0, 1]$ exists with

$$\mu_{A\wedge B}(u) = f(f^{-1}(\mu_A(u)) + f^{-1}(\mu_B(u))).$$

If f is a rational function in $\mu_A(u)$ and $\mu_B(u)$ then the only possible operator is that shown in definition 20. For γ=1, this reduces to the algebric product!

definition 2-13 (Hamacher-union-operator): The union of two fuzzy sets A and B is defined as

$$A \cup B = (u,\ \mu_{A\cup B}(u) \mid u \in U)$$

where

$$\mu_{A\cup B}(u) = \frac{(\gamma' - 1)\mu_A(u)\mu_B(u) + \mu_A(u) + \mu_B(u)}{1 + \gamma'\mu_A(u)\mu_B(u)},\ \gamma' \geq -1$$

For γ'=0 the Hamacher-union-operator reduces to the algebric sum.

definition 2-14 (Yager-intersection-operator): The intersection of two fuzzy sets A and B is defined as

$$A \cap B = (u,\ \mu_{A\cap B}(u) \mid u \in U)$$

where

$$\mu_{A\cap B}(u) = 1 - \min(1,\ ((1-\mu_A(u))^p + (1-\mu_B(u))^p)^{1/p})\ ,\ \ p \geq 1$$

definition 2-15 (Yager-union-operator): The union of two fuzzy sets A and B is defined as

$$A \cup B = (u,\ \mu_{A\cup B}(u) \mid u \in U)$$

where

$$\mu_{A\cup B}(u) = \min(1,\ (\mu_A(u)^p + \mu_A(u)^p)^{1/p})\ ,\ p \geq 1$$

The Yager-intersection operator converges to the min-operator (see def. 10.) for $p \to \infty$ and becomes the "bold-intersection" of definition 15 for p = 1 and, inversely, for $p \to \infty$ the Yager-union converges to the max operator (see def. 2-2.) and for p = 1 it becomes the "bold-union" of definition 2-8. Both operators satisfy the DeMorgan laws, and are commutative, associative for all p, monotonically nondecreasing in μ(u), and include the classical cases of dual logic. (But they are not distributive!)

Finally, we present another parametrized family of aggregation operators, which are commutative and associative, were published by Dubois and Prade [8,9]:

Definition 2-16 (Dubois-intersection): The intersection of two fuzzy sets A and B is defined as

$$A \cap B = (u,\ \mu_{A\cap B}(u) \mid u \in U)$$

where

$$\mu_{A\cap B}(u) = \frac{\mu_A(u)\cdot\mu_B(u)}{\max(\mu_A(u),\ \mu_B(u),\ \alpha)}\ ,\ \alpha \in [0,\ 1]$$

This intersection operator is decreasing with respect to α and lies between $\max\{\mu_A(u),\ \mu_B(u)\}$ (which is the resulting operation for $\alpha = 0$) and the algebric product $\mu_A(u)\cdot\mu_B(u)$ (for $\alpha = 1$). The parameter α is a kind of threshold since the following relationships hold for the defined intersection operation:

$$\mu_{A\cap B}(u) = \min(\mu_A(u),\ \mu_B(u)) \qquad \text{for} \quad \mu_A(u),\ \mu_B(u) \in [\alpha,\ 1]$$

$$\mu_{A\cap B}(u) = \frac{\mu_A(u)\cdot\mu_B(u)}{\alpha} \qquad \text{for} \quad \mu_A(u),\ \mu_B(u) \in [0,\ \alpha]$$

Definition 2-17 (Dubois-union): The union of two fuzzy sets A and B is defined as

$$A \cup B = (u, \mu_{A\cup B}(u) \mid u \in U)$$

where

$$\mu_{A\cap B}(u) = \frac{\mu_A(u) + \mu_B(u) - \mu_A(u)\cdot\mu_B(u) - \min(\mu_A(u),\mu_B(u),(1-\alpha))}{\max((1-\mu_A(u)),(1-\mu_B(u)),\alpha)}\ ,\ \alpha \in [0,1]$$

All the operators was mentioned so far include the case of dual logic as special case. The question may arise: Why are there unique definitions for intersection (= and) and union (= or) in dual logic and traditional set theory and so many suggested definitions in fuzzy set theory? The answer is simply that many operators (for instance product and min-operator) perform in exactly the same way if the degrees of membership are restricted to the values 0 or 1. If this is not longer requested they lead to different results.

1.2.4. Avaraging operators

A straightforward approach for aggregating fuzzy sets, for instance, in the context of decision making would be to use the aggregating procedures frequently used in utility theory or multi-criteria decision theory. They realise the trade-offs between conflicting goals when compensation is allowed, and the resulting trade-offs lie between the most optimistic lower bound and the most pessimistic upper bound, that is, they map between the minimum and the maximum degree of membership of the aggregated sets. Therefore they are called averaging operators. Operators such as the wegihted and unweighted arithmetic or geometric mean are examples of nonparametric averaging operators. In fact, they are adequate models for human aggregation procedures in decision environments and have empirically performed quite well [12].

The fuzzy aggregation operators "fuzzy and" and "fuzzy or" are suggested to combine the minimum and maximum operator [10,11], respectively, with arithmetic mean. The combination of these operators leads to very good results with respect to empirical data and allows compensation between the membership values of the aggregated sets [13].

Definition 2-18 (the "fuzzy and" operator): The "fuzzy and" operator is defined as

$$\mu_{and}(\mu_A(u),\ \mu_B(u)) = \gamma \cdot \min(\mu_A(u),\ \mu_B(u)) + \frac{(1-\gamma)(\mu_A(u)+\mu_B(u))}{2}\ ,\ u \in U,\ \gamma \in [0,\ 1]$$

Definition 2-19 (the "fuzzy or" operato)r: The "fuzzy or" operator is defined as

$$\mu_{or}(\mu_A(u),\ \mu_B(u)) = \gamma \cdot \max(\mu_A(u),\ \mu_B(u)) + \frac{(1-\gamma)(\mu_A(u)+\mu_B(u))}{2}\ ,\ u \in U,\ \gamma \in [0,\ 1]$$

The parameter γ indicates the degree of nearness to the strict logical meaning of "and" and "or", respectivelly. For γ=1 the "fuzzy and" becomes the minimum operator, the "fuzzy or" reduces to the maximum operator. γ=0 yields both the arithmetic mean.

Additional averaging aggregation procedures are symmetric summation operators, which as well as the arithmetic or geometric mean operators indicate some degree of compensation but in contrast to the latter are not associative. Examples of symmetric summation operators are the operators M1, M2 and N1, N2, known as symmetric summations and simmetric differences, respectively. Here the aggregation of two fuzzy sets A and B is pointwise defined as follows:

$$M_1(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)+\mu_B(u)-\mu_A(u)\cdot\mu_B(u)}{1+\mu_A(u)+\mu_B(u)-2\mu_A(u)\cdot\mu_B(u)}$$

$$M_2(\mu_A(u),\ \mu_B(u)) = \frac{\mu_A(u)\cdot\mu_B(u)}{1-\mu_A(u)-\mu_B(u)+2\mu_A(u)\cdot\mu_B(u)}$$

$$N_1(\mu_A(u),\ \mu_B(u)) = \frac{\max(\mu_A(u),\ \mu_B(u))}{1+|\mu_A(u)-\mu_B(u)|}$$

$$N_2(\mu_A(u),\ \mu_B(u)) = \frac{\min(\mu_A(u),\ \mu_B(u))}{1-|\mu_A(u)-\mu_B(u)|}$$

The above mentioned averaging operators indicate a "fix" compensation between the logical "and" and the logical "or". In order to describe a variety of phenomena in decision situations, several operators with different compensations are necessary. The operator that is more general in the sense that the compensation between intersection and union is expressed by a parameter γ was suggested by Zimmermann and Zysno[14] under the name "compensatory and".

Definition 2-20 (the "compensatory and" operator): The "compensatory and" operator is defined as

$$\mu_{AcompB}(u) = (\mu_A(u) \cdot \mu_B(u))^{(1-\gamma)} \cdot (1-(1-\mu_A(u)) \cdot (1-\mu_B(u)))^{\gamma}, \quad u \in U, \gamma \in [0, 1]$$

This "γ-operator" is obviously a combination of the algebric product (modeling the logical "and") and the algebric sum (modeling the logical "or"). It is pointwise injective, (except at zero and one) cintinuous, monotonous, and commutative. It also satisfies the DeMorgan laws and is in accordance with the truth tables of dual logic. The parameter indicates where the actual operator is located between the logical "and" and "or".

Other operators following the idea of parametrized compensation are defined by taking linear convex combinations of noncompensatory operators modelling the logical "*and*" and "*or*".

Definition 2-21 (combination aggregation for fuzzy sets): The aggregation of two fuzzy sets A and B by the convex combination between the min- and max-operator is defined as:

$$\mu_1(\mu_A(u), \mu_B(u)) = \gamma \cdot \min(\mu_A(u), \mu_B(u)) + (1-\gamma) \cdot \max(\mu_A(u), \mu_B(u)), \quad u \in U, \gamma \in [0, 1]$$

Combining the algebric product and algebric sum we obtain the following operation:

$$\mu_2(\mu_A(u), \mu_B(u)) = \gamma \cdot \mu_A(u) \cdot \mu_B(u) + (1-\gamma) \cdot (\mu_A(u) + \mu_B(u) - \mu_A(u) \cdot \mu_B(u)), \quad u \in U, \gamma \in [0, 1]$$

The relationships between different aggregation operators for aggregating two fuzzy sets A and B with respect to the three classes of *t*-norms, *t*-conorms, and averaging operators are represented in figure 5.1.

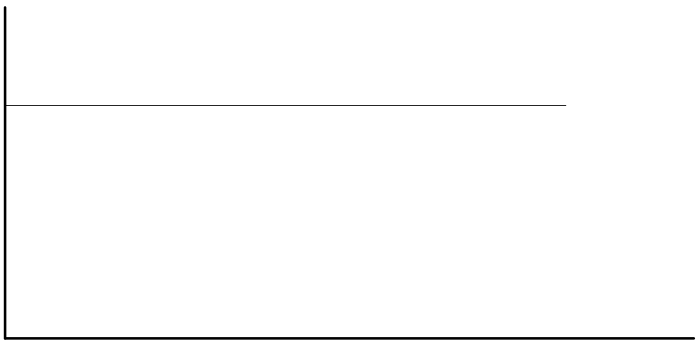

Figure 5.1. Mapping of t-norms, t-conorms, and averaging operators.

1.2.5. Criteria for Selecting Appropriate Aggregation

The numerous operators existing for aggregation of fuzzy sets might be confusing and might make it difficult to decide which one to use in a specific model or situation. Which rules can be used for such decision?

The following eight important criteria according to which operators can be classified are not quite disjunct; hopefully, they may be helpful in selecting the appropriate connective.

C1: Axiomatic Strength: We have listed the axioms that different authors wanted their operators satisfy. Obviously, everything else being equal, an operator is better the less limiting are the axioms it satisfies.

C2: Empirical Fit: If fuzzy set theory is used as modelling language for real situations or systems, it is not only important that the operators satisfy certain axioms or have certain formal qualities (such as associativity, commutativity), which are certainly of importance from a mathematical point of view, but the operators must also be appropriate models of real-system behaviour; and this can normally be proven only by empirical testing.

C3: Adaptability: It is rather unlike that the type of aggregation is independent of the context and semantic interpretation, that is, whether the aggregation of fuzzy sets models a human decision, a fuzzy controller, a medical diagnostic system, or a specific inference rule in fuzzy logic. If one wants to use a very small number of operators to model many situations, then these operators have to be adaptable to the specific context. This can, for instance, be achieved by parametrization. Thus min- and max-operators cannot be adapted at all. They are acceptable in situations in which they fit and under no other circumstances. (Of course, they have other advantages, such as numerical efficiency). By contrast, Yager's operators or the γ-operator can be adapted to certain contexts by setting the p's and γ's appropriatelly.

C4: Numerical Efficiency: Comparing the min-operator with, for instance, Yager's intersection operator or the γ-operator it becomes quite obvious, that the latter two require considerable more computational effort than the former one. In practice, this might be quite important, in particular when large problems have to be solved.

C5: Compensation: The logical "and" does not allow for compensation at all, that is, an element of the intersection of two sets cannot compensate a low degree of belonging to one of the intersected sets by a higher degree of belonging to another of them; in (dual) logic one can not compensate by higher truth of one statement for lower truth of another statement when combining them by "*and*". By compensation, in the context of aggregation operators for fuzzy sets, we mean the following:

Given that the degree of membership to the aggregated fuzzy set is

$$\mu_{\mathrm{Agg}}(u_k) = f(\mu_A(u_k),\ \mu_B(u_k)) = k$$

f is compensatory if $\mu_{\mathrm{Agg}}(u_k) = k$ is obtainable for a different $\mu_A(u_k)$ by a change in $\mu_B(u_k)$. Thus the min operator is not compensatory while the product operator, the γ-operator, and so forth are.

C6: Range of Compensation: If one would use a convex combination of min- and max-operator, a compensation could obviously occur in the range between min and max. The product

operator allows compensation in the open interval (0, 1). In general, the larger the range of compensation the better the compensatory operator.

C7: Aggregating Behavior: Considering normal or subnormal fuzzy sets, the degree of membership in the aggregated set depends very frequently on the number of sets combined. Combining fuzzy sets by the product operator, for instance, each additional fuzzy set "added" will normally decrease the resulting aggregate degrees of membership. This might be a desirable feature, it might, however, also not adequate.

C8: Required Scale Level of Membership Functions: The scale level (nominal, interval, ratio, or absolute) on which membership information can be obtained depends on a number of factors. Different operators may require different scale levels of membership information to be admissable. (For instance, the min-operator is still admissible for ordinal information while the product operator, strictly speaking, is not!) In general, again all else being equal, the operator that requires the lowest scale level is the most preferable from the point of view of information gathering.

1.3. Fuzzy Measures and Measures of Fuzzyness

1.3.1. General Discussion

The fuzzy set provides us with an intuitively pleasing method of representing one form of uncertainty. Consider, however, the jury members for a criminal trial who are uncertain abaut the guilt or innocence of the defendant. The uncertainty in this situation seems to be of different type; the set of people, who are guilty of the crime and the set of innocent people are assumed to have very distinct boundaries. The concern, therefore, is not with the degree to which the defendant is guilty but with the degree to which the evidence proves his or her membership in either the crisp set of guilty people or in the crisp set of innocent people. We assume that perfect evidence would point to full membership in one and only one of these sets. Our evidence, however, is rarely, if ever, perfect, and some uncertainty usually prevails. In order to represent this type of uncertainty, we could assign a value to each possible crisp set to which the element in question might belong. This value would indicate the degree of evidence or certainty of the element's membership in the set. Such a representation of uncertainty is known as a *fuzzy measure*. Note how this method differs from the assignment of membership grades in fuzzy sets. In the latter case, a value is assigned to each element of the universal set signifying its degree of membership in particular set with unsharp boundaries. The fuzzy measure, on the other hand, assigns a value to each crisp set of the universal set signifying the degree of evidence or belief that particular element belongs in the set. [15]

A fuzzy measure is thus defined by a function

$$g : \mathcal{P}(U) \rightarrow [0, 1],$$

which assigns to each crisp subset of U a number in the unit interval [0,1]. When this number is assigned to a subset $A \in \mathcal{P}(U)$, $g(A)$ represents the degree of the available evidence or our belief

that a given element of U (a priori nonlocated in any subset of U) belongs to the subset A. The subset to which we assign the highest value represents our best guess concerning the particular element in question. For instance, suppose we are trying to diagnose an ill patient. In simplified terms, we may be trying to determine whether this people belongs to the set of people with, say, pneumonia, bronchitis, emphysema, or common cold. A physical examination may provide us with helpful yet inconclusive evidence. Therefore we might assign a value, say 0.75, to our best guess, bronchitis, and a lower value to other possibilities such as 0.45 to pneumonia, 0.3 to a common cold, and 0 to emphysema. These values reflect the degree to which the patient's symptoms provide evidence for one disease rather than another, and the collection of these values constitutes a fuzzy measure representing the uncertainty or ambiguity associated with several well-defined alternatives. It is important to understand how this type of uncertainty is distinct from the vaguness or lack of sharp boundaries that is represented by the fuzzy set.

The difference between these two types of uncertainty is also exhibited in the context of scientific observation or measurement. Observing attributes such as type of cloud formation in meteorology, a characteristic posture of an animal in ethology, or a degree of defect of tree in forestry clearly involves situations in which it is not possible to drow sharp boundaries; such observations or measurements are inherently vague and consequently, their connection with the concept of the fuzzy set is suggestive. In most measurements in physics, on the other hand, such as the measurement of length, weight, electric current, or light intensity, we define classes with sharp boundaries. Given a measurement range, usually represented by an interval of real numbers [a, b], we partition this interval into disjoint subintervals

$$[a, a_1), [a_1, a_2), [a_2, a_3), \ldots, [a_{n-1}, b]$$

according to the desired (or feasible) accuracy. Then, theoretically, each observed magnitude fits exactly into one of the intervals. In practice, however, this would be warranted only if no observational errors were involved. Since measurement errors are unavoidable in principle, each observation that coincides with or is in close proximity to one of the boundaries $a_1, a_2, \ldots, a_{n-1}$ between two neighboring intervals involves uncertainty regarding its membership in the two crisp intervals (crisp subsets of the set of real numbers). This uncertainty clearly has all the characteristics of fuzzy measure.

1.3.2. Axiomatic System of Fuzzy Measures

In order to qualify as a fuzzy measure, the function g was, mentioned above, must have certain properties. These required properties were tradionally assumed to be the usual axioms of probability theory (or probability measures) [16, 17, 18].

Definition 3-1 (fuzzy measure): A set function $2g$ defined on $\mathcal{P}(U)$ which has the following properties is called a fuzzy measure:

m1: $g(0) = 0, g(\mathrm{U}) = 1$ (boundary conditions)

m2: If $\mathrm{A}, \mathrm{B} \in \mathcal{P}(U)$ and $\mathrm{A} \subseteq \mathrm{B}$, then $g(\mathrm{A}) \leq g(\mathrm{B})$ (monotonicity)

m3: If $A_n \in \mathcal{P}(U)$, $A_1 \subseteq A_2 \subseteq A_3 \ldots$ or $A_1 \supseteq A_2 \supseteq A_3 \ldots$ (continuity)
(i.e., the sequence is monotonic,) then $\lim_{n\to\infty} g(A_n) = g(\lim_{n\to\infty} A_n)$.

Axiom m1 states that despite our degree of evidence, we always know that the element in question definitely does not belong to the empty set and definitely does belong to the universal set. The empty set, by definition, does not contain any element, hence it cannot contain the element of our interest either; the universal set, on the other hand, contains all elements under consideration in each particular context and, therefore, it must contain our element as well.

Axiom m2 requires that the evidence of the membership of an element in a set must be at least as great as the evidence that the element belongs to any subset of that set. Indeed, when we know with some degree of certainty that the element belongs to a set, then our degree of certainty that it belongs to a larger set containing the forner set can be greater or equal, but it cannot be smaller.

Axiom m3 is clearly applicable only to an infinite universal set. It can, therefore, be disregarded when we are dealing with a finite universal set. The axiom requires that for every infinite sequence A_1, A_2,... of nested (monotonic) subsets of *X* that converge to the set

$$A = \lim_{n\to\infty} A_n,$$

the sequence of numbers $g(A_1)$, $g(A_2)$, ... must converge to the number $g(A)$. That is, the axiom requires that g is a continuous function.This axiom can also be viewed as a requirement of consistency: calculation of $g(A)$ in two different ways, either as the limit of $g(A_n)$, for $n\to\infty$ or by application of the function g to the limit of A_n for $n\to\infty$, is required to yield the same value.

Definition 3-2 (λ-fuzzy measure): By a λ-fuzzy measure, g_λ, defined in $B \subseteq \mathcal{P}(U)$, we mean a fuzzy measure for which, for each A and B such that $A \cap B = 0$ and $A \cup B \in B$, we get

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda \cdot g_\lambda(A) \cdot g_\lambda(B)$$

where $\lambda \in (-1, \infty)$. This fuzzy measure is called a λ-additive one. However, the fuzzy measure g_λ in B is countably λ-additive if, for each sequence of $\{A_n\}$ pairwise disjoint sets in B such that their union belongs to B, one has

$$g_\lambda(\cup A_n) = \frac{1}{\lambda}\left[\Pi(1+\lambda \cdot g_\lambda(A_i)) - 1\right]\text{Me'g hiba's:produktum: i=1...}\infty\text{!!!}$$

Note that, for $\lambda = 0$, g_λ converts into a probability measure

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B).$$

Some interesting properties of the λ-fuzzy measures are listed below:

1. if $\lambda \geq 0$, then $g_\lambda(A \cup B) \geq g_\lambda(A) + g_\lambda(B)$.

2. for a difference of two set A and B

$$g_\lambda(A / B) = \frac{g_\lambda(A) - g_\lambda(B)}{1 + \lambda \cdot g_\lambda(B)},\ B \subset A$$

so, for a complement of A, one gets

$$g_\lambda(\overline{\mathrm{A}}) = \frac{1 - g_\lambda(\mathrm{A})}{1 + g_\lambda(\mathrm{A})}$$

Definition 3-3 (λ-fuzzy measure in a finite space X): The λ-fuzzy measure can be conveniently applied in a finite space X, say $\mathrm{X} = \{\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_n\}$. Let us be given values of the fuzzy measure attached to each x_i equal to $g_\lambda(\{\mathrm{x}_i\})$. Then, for each $F \subset X$, its fuzzy measure is equal to

$$g_\lambda(\mathrm{F}) = \frac{1}{\lambda}\left[\Pi(1 + \lambda \cdot g_\lambda(\mathrm{x}_i)) - 1\right] \text{ (Kapcsos zarojelek!)}$$

Thus, for a set F consisting of two elments x_1 and x_2, the above relationship reduces to the form

$$g_\lambda((\mathrm{x}_1, \mathrm{x}_2)) = g_\lambda((\mathrm{x}_1)) + g_\lambda((\mathrm{x}_2)) + \lambda \cdot g_\lambda((\mathrm{x}_1)) \cdot g_\lambda((\mathrm{x}_2)). \text{ (Kapcsos zarojelek!)}$$

On the basis of the boundary condition, $g_\lambda(\mathrm{X}) = 1$, the value of the parameter λ can be derived by solving an algebraic equation of the first order

$$1 = g_\lambda((\mathrm{x}_1)) + g_\lambda((\mathrm{x}_2)) + \lambda \cdot g_\lambda((\mathrm{x}_1)) \cdot g_\lambda((\mathrm{x}_2)). \text{ (Kapcsos zarojelek!)}$$

i.e.

$$\lambda = \frac{1 - g_\lambda((\mathrm{x}_1)) - g_\lambda((\mathrm{x}_2))}{g_\lambda((\mathrm{x}_1)) \cdot g_\lambda((\mathrm{x}_2))}. \text{ (Kapcsos zarojelek!)}$$

For a finite space X, the value of λ can be obtained, for instance, by the Newton-Raphson iterative scheme, namely, the value of this parameter for the k+1th iteration is equal to

$$\lambda(\mathrm{k}+1) = \lambda(\mathrm{k}) - \frac{\mathrm{F}(\lambda(\mathrm{k})) - 1}{\mathrm{F}'(\lambda(\mathrm{k}))}$$

where

$$\mathrm{F}(\lambda(\mathrm{k})) = \frac{1}{\lambda}\left[\Pi(1 + \lambda(\mathrm{k}) \cdot g((\mathrm{x}_k)) - 1)\right]$$

and

$$\mathrm{F}'(\lambda(\mathrm{k})) = \frac{1}{\lambda}\left[(1 + \lambda(\mathrm{k}) \cdot \mathrm{F}(\lambda(\mathrm{k}))\right] \cdot \sum_{k=1}^{n} \left(\frac{g((\mathrm{x}_k))}{1 + \lambda(\mathrm{k}) g_\lambda((\mathrm{x}_k))} - \mathrm{F}(\lambda(\mathrm{k}))\right).$$

Definition 3-4 (fuzzy integral): By a fuzzy integral of h with respect to a fuzzy measure $g_\lambda(\cdot)$ over a set A $\int_A \cdot g_\lambda(\cdot)$ we mean a non-negative number equal to

$$\int_A \cdot g_\lambda(\cdot) = \sup_{\alpha \in [0,1]}\left[\alpha \wedge g(A \cap F_\alpha)\right]$$

with F_α being an α-cut of h, $F_\alpha = \{u \in U | h(u) \geq \alpha\}$.

The fuzzy integral is a nonlinear functional processing some properties of an ordinal integral; the most specific ones are listed below:

i1: $\int_A c \cdot g_\lambda(\cdot) = c$

i2: $\int_A (c \wedge h) \cdot g_\lambda(\cdot) = c \wedge \int_A h \cdot g_\lambda(\cdot) =$

i3: $\int_A (h_1 \cup h_2) \cdot g_\lambda(\cdot) \geq \int_A h_1 \cdot g_\lambda(\cdot) \wedge \int_A h_2 \cdot g_\lambda(\cdot)$

i4: $\int_A (h_1 \cap h_2) \cdot g_\lambda(\cdot) \leq \int_A h_1 \cdot g_\lambda(\cdot) \wedge \int_A h_2 \cdot g_\lambda(\cdot)$

Computations of the fuzzy integral for a finite U can be performed after preliminary operations. They rely on an arrangement of the values of the membership function h in non-increasing order, say

$$h(x_1) \geq h(x_2) \geq \ldots h(x_n)$$

where

$$\int_A h \cdot g_\lambda(\cdot) = \max_{1 \leq i \leq n} \left[h(u_1) \wedge g_\lambda(X_i) \right]$$

where X_i consists of a set of ith, the greatest elements of X, $X_i = \{x_1, x_2, \ldots, x_i, \}$.

Note that $g_\lambda(x_i)$ forms a non-decreasing sequence of numbers, so in fact the value of the fuzzy integral is taken as the height of the intersection $h(x_1) \wedge g_\lambda(X_i)$.

In the framework of fuzzy set theory Zadeh introduced the notion of a *possibility distribution* and the concept of possibility measure, which is a special type of the fuzzy measure. A possibility measure is defined as follows:

Definition 3-5 (possibility measure): A possibility measure is a function p: P(U) $\rightarrow$ [0,1] with the properties

p1: $p(0) = 0$ and $p(X) = 1$

p2: If A, B $\in$ P(U) and A $\subseteq$ B, then $p(A) \leq p(B)$

p3: $p(\bigcup_{i \in I} A_i) = \sup_{i \in I} p(A_i)$ with an index set I

It can be uniquelly determined by a possibility distribution function

$$f: U \rightarrow [0,1] \quad \text{by} \quad p(A) = \sup_{x \in A} f(x), \quad A \subset U.$$

It follows directly that f is defined by $f(x) = p(\{x\}) \quad \forall x \in U$.

A possibility is not necessarily a fuzzy measure, it is, however, a fuzzy measure, if U is finite and if the possibility distribution is normal,-that is, mapping into [0, 1].

1.3.3. Measures of Fuzziness

Measures of fuzziness by contrast to fuzzy measures try to indicate the degree of fuzziness of a fuzzy set. A number of approaches to this end have bacome known []. We shall, as an illustration, discuss two of those measures. Suppose for both cases, that the support af A is finite.

1.3.3.1. The "Entropy" of Fuzzy Sets

Let μA(*u*) be the membership function of the fuzzy set A for $u \in$ U, U finite. It seems plausible that the measure of fuzziness d(A) should then have the following properties[]:

p1. d(A) = 0 if A is a crisp set in U.

p2. d(A) assumes a unique maximum if $\mu_A(u) = 1/2 \;\; \forall u \in U$.

p3. d(A) ≥ d(A') if A' is "crisper" than A, i.e., if $\mu_A(u) \le \mu_{A'}(u)$ for $\mu_A(u) \le 1/2$ and $\mu_A(u) \ge \mu_{A'}(u)$ for $\mu_A(u) \ge 1/2$.

p4. $d(\overline{A}) = d(A)$ where $\overline{A}$ is the complement of A.

In the above case a measure of the fuzziness the "entropy" of a fuzzy set, what is defined as follows:

Definition 3-6 (the entropy as a measure of fuzziness): The entropy as a measure of a fuzzy set

A = {(*x*, $\mu_A(x)$)} is defined as

$$d(A) = H(A) + H(\overline{A}),$$

$$H(A) = -K\sum_{i=1}^{n} \mu_A(u_i) \cdot \ln(\mu_A(u_i)), \quad u \in U$$

where *n* is the number of elements in the support of A and *K* is a positive constant.

Using Shannon's function $S(x) = -x\ln x - (1-x)\ln(1-x)$, the expression can be simplified to the following form:

$$d(A) = -K\sum_{i=1}^{n} S(\mu_A(u_i))$$

1.3.3.2. The Complement Distance of Two Fuzzy Sets

If A is a fuzzy set in U and and $\overline{A}$ its complement, then by contrast to crisp sets, it is not necesseraly true that

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = 0$$

This means that fuzzy sets do not always satisfy the law of the excluded middle, which is one of their major distinctions from traditional crisp sets. That is, we have to define the distance between a fuzzy set and its complement, as a possible metric[].

Definition 3-7 (complement distance):

$$\mathrm{D_p(A,\overline{A})} = (\sum_{i=1}^{n} \left|\mu_A(u_i) - \mu_{\overline{A}}(u_i)\right|^p)^{1/p}, \quad p = 1,2,3,...$$

Let S = supp($\overline{\mathrm{A}}$): $D_p(S,\overline{S}) = \|S\|^{1/p}$

Definition 3-8 (Yager-Measure of fuzziness): A measure of the fuzziness af A can be defined as

$$f_p(\mathrm{A}) = 1 - \frac{D_p(A,\overline{A})}{\|\mathrm{supp}(\overline{A})\|}$$

So $f_p(A) \in [0,1]$. This measure also satisfies properties p1 to p4 were described above. For p=1, $D_p(A,\overline{A})$ yields the Hamming metric

$$D_p(A,\overline{A}) = \sum_{i=1}^{n} \left|\mu_A(u_i) - \mu_{\overline{A}}(u_i)\right|.$$

Because 8, this becomes

$$D_p(A,\overline{A}) = \sum_{i=1}^{n} \left|2\mu_A(u_i) - 1\right|$$

For p=2, we arrive at the Euclidean metric

$$D_p(A,\overline{A}) = (\sum_{i=1}^{n} \left|\mu_A(u_i) - \mu_{\overline{A}}(u_i)\right|^2)^{1/2}$$

and for $\mu_{\overline{A}}(u) = 1 - \mu_A(u)$, we have

$$D_2(A,\overline{A}) = (\sum_{i=1}^{n} (2\mu_A(u_i) - 1)^2)^{1/2}$$

As it is coming from the above definitions, the complement of a fuzzy set is not uniquely defined. It is therefore not surprising that for other definitions of the comlement and for other measures of distance, other measures of fuzziness will result, even though they all focus on the distinction between a fuzzy set and its complement.