2. Quantitative modelling

2.1 Fundamental laws of quantitative modelling

The quantitative models, namely the mathematical models describe the space-time behaviour of processes. The general mathematical description of processes usually consists of coupled systems of algebraic and differential equations. The models describe systems in terms of model variables classified as inputs U_i, state variables X_i, outputs Y_i, and model parameters P_i.

In creating mathematical models of processes it is particularly important that the model nearly always consists of parts, the so-called physico-chemical part and the apparatus specific part [1].
The physico-chemical component of the model includes relations. The validity of relations used in the physico-chemical components of the model is totally independent of the particular apparatus in which the model operations take place. This self consistency of physical component insures that the model can be applied to a variate of apparatus, which is a great advantage considering the wide range of different solution algorithm of differential equations, or software packages for engineering applications, numerical methods and computing capacity of computers.
Typical processes taking place in continuos-flow equipment so obey balance equations of conserved quantities.
Considering a continuous process using mathematical model of it we generally interested in the normal, periodic behaviour of the process, or in the transient of the process taking place in special circumstances, caused by starting, stopping of the process or generated by some type of disturbances.
Mathematical model can be dedicated to the proper case setting initial conditions of system variables and inputs, forming other apparatus specific equations. Initial setting of process parameters and histories of them may be added.
For demonstration this structure of mathematical process model Polke gives mathematical model of a destillation columns.

Physico-chemical part (state relations)

Enthalpy	$J = J(x, T)$
Density	$\varsigma = \varsigma(x, T)$
Ideal gas law	$p * V = v * R * T$

Apparatus-specified part (process equations)

Balance equations $\frac{\partial x}{\partial T} + \mathrm{div}(\rho \vec{v}, x)$

= diffusion term + sources - sinks

″Engineering formulas″

Initial and boundary conditions

In the physico-chemical part the equilibrium relations are used between the mole fractions in gas and liquid the relation for the boiling point line (which interrelates the pressure with the molar fraction of liquid and temperature), the ideal gas law, and relations for the liquid density, liquid enthalpy, heat variation, and (if appropriate) reaction rates.
The apparatus component of the model consists of the material and heat balances for condenser, trays, bottom and vaporiser, the separating effect of the trays (the Murphree efficiency E), the pressure balances for determining the pressure drop across individual trays (using basic geometrical data such as perforation diameter, tray thickness, and active tray surface area) and finally a weir relation to give the down flowing liquid flow rate (again based on geometrical data such as weir height).
As this concrete example shows for creating a proper model of a single part (a distillation columns) of an industrial process several laws of different part of physics had to be applied.
The feature of laws used in modelling depends heavily on the purpose of the process. E.g. an electrical motor typically represented in an electrical circuit by its resistance, inductance and capacitance, and the apparatus specific part laws come from the field of electrical engineering. The same electrical motor taking part with its mechanical parameters in a model for designing the shape, size and mechanical properties of a coffee-grinder.
In the next part we list only the most general equations of physics, first of all the generally used equations of conserved quantities, as the momentum, the mass and the enthalpy (heat balance).

General definition of energy

Energy is a generally used concept in most field of physics and can be applied as capability for doing work. The main feature of energy is the fact, that it is conserved.
The total energy of an isolated system is conserved.
This statement means: energy cannot increase in one place without correspondingly decreasing somewhere else.
Energy plays very powerful problem-solving role in almost all fields of physical problems. Energy has no direction, it can be represented by an ordinary - scalar - number, it is easier to deal with than vectors.
The principle of conserved energy can be represented in different forms using the proper phenomena of different field of physics.

Mass conservation

The mass is a form of energy. A system of mass M contains a total energy $E=M* c^2$, where c is the speed of light. Although this relation is not detectable in case of daily life object, the conservation of mass is a consequence of this relation.
An object or a system keeps constant mass even if drastic changes (such as chemical reactions) are allowed to take place inside the object or the system.

Mechanical energy

Mechanical energy means two kinds of energy, first associated with a particle's altitude - gravitational, potential energy -, and with a particle's motion - the kinetic energy-.
Potential energy /U/ (in case of uniform gravity /g/) proportional to the mass /m/ and the altitude /z/ of the object:

$U = m * g * z$ [joule].

Kinetic energy /K/ - energy of motion depends on the mass and the speed /v/ of the object:

$$K = \frac{1}{2} * m * v^2 \quad [\text{ joule }].$$

Total mechanical energy of an object:

$E = U + K$,

and the conservation of mechanical energy:

$$(U + K)_f = (U + K)_i$$

The work /W/ on an object

$W = \mathbf{F} * l$ [joule],

where **F** is the constant force (with direction) and l is the displacement.
Expressed for infinitesimal small part of this, when the force are not constant or the displacement have different directions:

$dW = \mathbf{F} * d\mathbf{r}$,

and for the whole work for the motion between A and B points

$$W = \int_a^b F \, dr$$

Supplementary work on a system changes the mechanical energy of it:

$$W_{sup} = \Delta (U + K).$$

Dealing with mechanical problems we have to take our considerations the Newton's laws.

Newton's First Law

The law of inertia asserts, that an isolated particle maintains a uniform motion.

Newton's Second Law

The law of acceleration /a/ defines the force: the total force vector, $\mathbf{F}_{tot}$ acting on a particle with mass m causes the acceleration of it, defined as:

$\mathbf{F}_{tot} = m * \mathbf{a}$ [newton].

Newton's Third Law

The law of action and reaction states that action and reaction forces are equal in magnitude and opposite in directions:

$$\mathbf{F}_{\text{AonB}} = -\mathbf{F}_{\text{BonA}}.$$

The forces have to be added as vectors, and the vector sum of forces operated on an object of zero acceleration is equal to zero.

$$\mathbf{F}_{\text{tot}} = 0.$$

Conservation of Momentum

As we seen in the case of energy, conserved quantities are the great help of solving physical problem even in case of complicated system.
In an isolated system the total momentum is also conserved, and this idea can be applied to several problems, e.g. collision, the examination of gas pressure, the force exerted by fluid stream, e.g. in rocket engine.
The centre of mass, in terms of which an object is legitimately represented as a particle.
Conservation of momentum partly follows from Newton's laws of motion, but it is more fundamental than these laws.
The momentum vector of a particle (of mass m and velocity **v**) is denoted by **p** and defined as

$$\mathbf{p} = m * \mathbf{v} \qquad [\text{kg}\frac{\text{m}}{\text{s}}] \ .$$

For a system of many particles the total momentum vector is:

$$\mathbf{p}_{\text{syst}} = m_1 * \mathbf{v}_1 + m_2 * \mathbf{v}_2 + \ldots = \sum_i m_i * \mathbf{v}_i \ , \quad [\text{kg}\frac{\text{m}}{\text{s}}]$$

The Newton's Second Law expanded for momentum:

$$\mathbf{F}_{\text{tot}} = m * \mathbf{a} = m * \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(m * \mathbf{v}\right) = \frac{d\mathbf{p}}{dt}$$

The total force on a particle equals the rate of change of its momentum.
If the system is not isolated, then the rate of change of the system's total momentum equals to the net external force acting on the system:

$$\frac{d\mathbf{p}_{\text{syst}}}{dt} = \mathbf{F}_{\text{ext}} = \mathbf{F}_{\text{tot}} \ .$$

Thus, a whole system can be legitimately be represented as a particle, at least as far as its momentum is concerned. The motion of this particle represents that of the whole system,

and the system can be considered to have its total mass concentrated at a point, called the centre of mass.
The position of this point:

$$r_c = \sum_i \frac{m_i * r_i}{m_{system}}, \quad \text{or}$$

$$r_c = \int \frac{r}{m_{system}} dm$$

for continuous objects.

The total momentum of the system is

$\mathbf{p}_{system} = m_{system} * \mathbf{v}_c$, and

$\Delta \mathbf{p}_{system} = 0$, if $\mathbf{F}_{ext} = 0$,

because of the conservation of momentum.

Rotational motion

Analogies between linear motion and fixed-axis rotation

Linear		Rotational	
Distance coordinate	x	Angular coordinate	###
Velocity	$v_x = \frac{dx}{dt}$	Angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a_x = \frac{dv_x}{dt}$	Angular acceleration	$\alpha = \frac{d\omega}{dt}$
Mass	m	Moment of inertia	$I = \sum_i m_i r_i^2$ (r=distance to axis)
Momentum	$p_x = mv_x$	Angular momentum	$L = I\omega$
Force	F_x	Torque	$\tau = rF_\perp = r_\perp F$
Kinetic energy	$K = \frac{1}{2} mv_x^2$	Kinetic energy	$K = \frac{1}{2} I\omega^2$
Work	$dW = F_x dx$	Work	$dW = \tau d\theta$
Power	$\frac{dW}{dt} = F_x v_x$	Power	$\frac{dW}{dt} = \tau\theta$

Laws of motion

	$F_{tot,x} = m * a_x$		$\tau_{tot} = I * \alpha$

	$\frac{dp_x}{dt} = F_{tot,x} = F_{ext,x}$ (=0 for isolated sys.)		$\frac{dL}{dt} = \tau_{tot} = \tau_{ext}$ (=0 for isolated sys.)

The momentum inertia / **L** / and the torque / τ / are the analogues of mass and force, and with these two ideas the rotational motion can be discussed similarly to the linear motion. The analogue of the momentum, the angular momentum is also conservative considering a rotational system.

The angular momentum conservation means:

$$\frac{d\mathbf{L}_{system}}{dt} = 0 \text{ , if } \tau_{ext} = 0.$$

On the other way

$$\frac{d\mathbf{L}_{system}}{dt} = \tau_{ext},$$

where τ_{ext} is the total external torque applied on the system.

Fluids

Pressure:

$$\mathbf{P} = \frac{\mathbf{F}}{A} \quad , \quad [1 \text{ pascal} = 1 \ \frac{\text{newton}}{m^2}] \ .$$

Density:

$$\varsigma = \frac{m}{V} \qquad [\ \frac{kg}{m^3}\ ] \ .$$

Archimedes's principle:

$w' = w - w_{water}$,

the buoyantly of w_{water} equals the weight of the displaced fluid.

Pascal's principle:

$$P_2 - P_1 = \varsigma * g * z.$$

In a uniform fluid pressure increases linearly with depth.

Temperature, heat transfer

Heat and the temperature are the basic ideas of thermal physics and the science of thermodynamics.
By the Charles' law under heating, the volume of the enclosed gas - at constant pressure - increases.

$T = a * V$ at constant pressure, or

$\frac{V_1}{T_1} = \frac{V_2}{T_2}$ any ideal gas at constant pressure.

Joule's experiment - heating with warming and with mechanical word - shows, that the mechanical equivalent of heat can be defined, and every heat transfer is an energy transfer. Thus the conversation of energy can be expanded to the heat transfer.

1 thermomechanical kilo calorie, 1 Kcal = 4184 joules exactly.

Heating of mass m through a small temperature interval ΔT the required heat to transfer is:

$Q = c * m \# \# \# \Delta T$ [joule],

where c is the substance's specific heat.

Avogadro's number is the number of atoms in 12 kg of Carbon 12, the most common isotope of carbon. Or, Avogadro's number, N_a is the number of atomic mass units /u/ in one kilogram:

$N_a = \frac{1kg}{1u}$ [dimensionless]

$N_a \# \# \# 6 * 10^{23}$.

Ideal-gas law (Boyle's law)

$P * V$ = constant, if T = constant.

Thermodynamics

The process, changing heat into work has a great practical importance considering engine of different principles being used in our industrial society.
The absolute limitations on the amount of work obtainable from heat source are given by the First and Second Laws of Thermodynamics.
The word output of a gas during the expansion equals to the area under the graph of P against V, which means:

$$W = \int_{V_1}^{V_2} P\, dV \ ,$$

where V_1 is the initial, V_2 is the final volume.
Apply heat transfer Q into a system, the possible work output, W, arising from the material's expansion and the stored thermal energy, U, the internal energy of the material itself have to be considered.
Conservation of total energy can be written in form:

$$Q = \Delta U + W \ .$$

This is the First Law of thermodynamics. Considering an infinitesimal heat input:

$$dQ = dU + dU = dU + P * dV \ .$$

The Second Law of Thermodynamics states:
There exist no process whose net result is a heat transfer from a cold to a hot object, without any other energy transfer.
In the case of cycle processes the first law has the next form:

$$W_{tot} = Q_{in} - Q_{out} \ .$$

The Second Law for cyclic processes:

$$\frac{Q_{out}}{T_-} = \frac{Q_{in}}{T_+}$$

The engine operate between the temperatures T_+ and T_- , Q_{in} comes from the external "hot source" at a constant T_+, Q_{out} is transferred into an external "cold source" at a lower constant T_- .
The maximum efficiency of a reversible engine:

$$n_{rev} = \frac{W_{tot}}{Q_{in}} = 1 - \frac{T_-}{T_+} \ ,$$

which can never reach 100%.
Considering the

$$\frac{Q_{in}}{T_+} - \frac{Q_{out}}{T_-} = 0 \text{ equation,}$$

where the first part can be seen as the increase in entropy of engine due to the heat input, and the second part as the decrease in entropy due to heat output. Thus, any system's entropy is conserved over a complete reversible cycle.
Considering the waste of energy during in natural and man-machine processes, and a wasteful process is an irreversible one.
An irreversible process increases the total existing entropy.

Elastic energy

Hook's Law :
The spring exerts a restoring force / F_x / proportional to its elongation or compression / x /:

$F_x = - k \# \# \# x$,

where k is the spring constant [newton / n].
The elastic potential energy

$U = k \# \# \# x^2$,

where x is the elongation, which is smaller then the elastic limit of the spring.
Total conserved energy of a harmonic oscillator, e.g. a spring / k / with a mass / m /:

$$E = U + K = \frac{1}{2} * k * x^2 + \frac{1}{2} * m * v^2,$$

where x is the maximum of elongation, v is the speed of the mass passing the equilibrium point.

Electrical energy

Conservation of total electric charge:
A system of initial charge Q will keep the system's total charge equal to Q. An isolated system with zero charge can never develop a net of charge.

$\Delta Q_{tot} = 0$.

The conservation of electrical energy is expressed also in Kirchoff's two law.
Voltage rule or loop rule states:

$V_1 + V_2 + \ldots = 0.$

The voltages, all measured in the same direction around a single loop of a circuit add up to zero.

Current rule, or node or junction rule:

$I_1 + I_2 + \ldots = 0$.

All currents flowing towards a common junction have zero sum.

Ohm's law states that in care of a resistance the current through the resistor proportional to V applied on it.

$I = \frac{V}{R}$, or $R = \frac{V}{I}$, [1 ohm = $\frac{1 \text{ volt}}{1 \text{ amper}}$].

Any load or appliance, connected into electric circuit, receives a power that is the product of the current through it times the voltage across its terminates.

P = V ### I , [1 watt = 1 volt ### 1 amper].

The energy is supplied may appear in various form, e.g. heat, mechanical motion, etc..
A resistor convert energy to heat:

$$P = R*I^2 = \frac{V^2}{R}.$$

In various modified forms, the dynamo is the most common industrial generator of electronic power. The electromotive force, the terminal voltage on a rotating coil in magnetic field is given as:

V = - N ### ###### B ### A ### cos ######t,

where N is the number of turns of the coil, w is the angular frequency of the rotation, B is the permanent magnetic field, A is the coil frame's area.
The relation between the primary and secondary alternating voltages of an iron-core transformer approximately is:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2},$$

where N_1 and N_2 are the turn numbers of coils, respectively.
Magnetic energy stored in a coil

$$W = \frac{1}{2}LW = \frac{1}{2}L * I^2,$$

where L is the coil's inductance, I is the current of the coil.
Electrostatic energy stored in a capacitor :

$$W = \frac{Q^2}{2}*C,$$

where Q is the charge of the capacitance, C is the capacity of it.

Energy equation of a parallel LC circuit

$$W_{tot} = W_l + W_c = \frac{{Q_o}^2*(\sin^2\omega t + \cos^2\omega t)}{2}*C,$$

where $\frac{Q_0^2}{2}*C$ is constant.

$f = \frac{\omega}{2\pi}$ is the natural frequency of the oscillation in the circuit, Q_o is the whole charge in the circuit. The energy is converted back and forth between electric and magnetic energy.

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2.3. Quantitative models of typical industrial processes

2.3.1. Static state of mass-flowing systems

At the first glance an industrial process can be seen as a simple mass-flowing system. The state of this system can be characterised by the currents of the mass and the intensity parameter, the pressure.
Simple resistance function of a mass-flowing device gives the relation between the volume of flowing mass and the pressure generating it.
With the help of loop equations and junction equations the current intensity and the volume of the pressure can be determined anywhere in the system.
The static state of a mass-flowing system can be totally characterised by the system of equations mentioned above.

Resistance functions of mass-flow devices

1. Pressure on the direct pipes

$$\Delta p = \frac{\lambda l}{2d} * \frac{1}{\varsigma A^2} * \Phi_m{}^2 = R(\Phi_m) * \Phi_m$$

where λ is the friction coefficient,
l the length of the pipe,
d the internal or equivalent diameter,
A the cross-section of the pipe,
ς the density of the flowing mass (depending on the pressure and the temperature),
Φ_m is the mass flowing through,
$R(\Phi_m)$ is the resistance function of the pipe.

2. Pipe junctions

$$\Delta p = \frac{\xi}{2\varsigma A^2} * \Phi_m{}^2$$

where ξ is the loss factor of the junction.

3. The Borda loss of reducing pipes

$$\Delta p = \frac{1}{2\varsigma A_1} * (1 - \frac{A_1}{A_2})^2 * \Phi_m{}^2 = R(\Phi_m) * \Phi_m$$

where A_1 and A_2 are the cross-sections causing the loss.

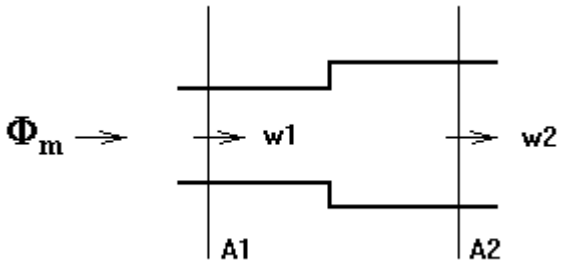


Figure no. 1. *The Borda loss*

4. Tube panel and rows of tubes

The pressure difference can be calculated by the relationship of pipe junctions. In this case A is the free flowing cross section. The ζ loss coefficient depends on the arrangement of pipes.
In the arrangement of the 2. figure, case a:

$$\zeta = (6+9\mathrm{m})*(\frac{s_1}{d})^{-0,23}*\mathrm{Re}^{-0,26}$$

In the case b, if

$\frac{s_1}{d} < \frac{s_2}{d}$, then

$$\zeta = (4+6,6\mathrm{m})*\mathrm{Re}^{-0,26}.$$

If $\frac{s_1}{d} > \frac{s_2}{d}$, then

$$\zeta = (5,4+3,4\mathrm{m})*\mathrm{Re}^{-0,28}$$

In these relations m is the number of pipe plans, Re is the Reynolds number relating on the flowing speed calculated on the free flowing area.

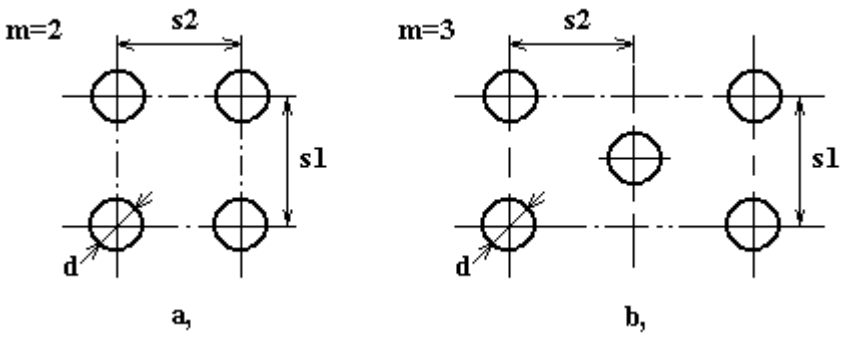


Figure no. 2. *Tube panel and rows of tubes*

5. Charged equipment

The pressure difference can be calculated by the equation we have seen in case of pipe junctions. Considering the loss coefficient we may have two cases:

a. Gas flowing through dry charge

$$\zeta = \frac{f' * H * \sigma}{V^3}$$

In the relation

$$f' = \frac{35}{Re_G}, \text{ if } Re_G \langle 40,$$

$$f' = \frac{4}{Re_G^{0.2}}, \text{ if } Re_G \rangle 40 .$$

The Reynolds number of the gas flowing through the charge :

$$Re_G = \frac{4w\varsigma}{\alpha\eta} .$$

The specific free surface is:

$$\sigma = \frac{A_v}{A} ,$$

where A_v is the real free surface of the charged pipe (m^2),
A is the cross-section of the empty equipment,
η and ς are the dynamic viscosity and density of the flowing mass,
H is the height of the charge in the direction of the flow (m),
V is the volume of the charge (m^3).

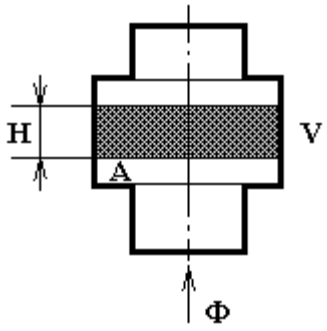


Figure no. 3. *Charged pipe*

b. The gas flowing through wet charge

$$\zeta = \frac{f' * H * \sigma}{V^3} * \Omega .$$

The relation is the extended version of the previous case, Ω represents the quality and the humidity of the charge.

Pressure sources

The pressure-increasing engines are the pumps, the ventilators and the compressors. The operations of these equipment also depend on the volume of flowing mass.

1. Volumetric displacement engines

In case of a volumetric displacement engine the transferred volume Φ_v is theoretically independent of the pressure difference being generated. The volumetric displacement engine creates such a big pressure difference, which is needed to maintain the flowing volume determinated by the main dimensions and the revolution number. The magnitude of the pressure is not defined, so the persistence of the volume-current can be used:

$$\Phi_v = \text{constant}.$$

Using current of mass:

$$\frac{1}{\varsigma} * \Phi_m = \text{constant}.$$

2. Rotational pumps

In case of permanent revolution number the relation between the pressure difference and the mass volume is the following:

$$\Delta p = a_1 + a_2 * \Phi_v + a_3 * \Phi_v^{\,2},$$

where the a coefficients depend on the parameters of the mass flow and relate to a dedicated engine.
In the practice we generally use the best analytical approximation of the real measured characteristic curve of the pressure generation instead of theoretical functions. The p curve based on mass currents is:

$$\Delta p = a_1 + a_2 * \frac{\Phi_m}{\varsigma} + a_3 * \frac{\Phi_m^{\,3}}{\varsigma},$$

where $\varsigma = \varsigma(p, \vartheta, \ldots)$.

2.3.2. Elementary units of technological processes

The most industrial process consists of several simple, elementary processes repeated one after the other. The elementary processes are performed by elementary units, which are connected by the mass flow and form the complex process.
In this chapter we will dealing with these elementary units and their processes. These can be seen as processes with concentrated parameters, so their dynamic behaviour can be represented by simple differential equations.

1. Simple storage unit

If the storage capacity of the unit is infinite, the actual store is W, the input flow is time dependent, but the output flow is not depend on the input, than the change of the store is:

$$dW = \left[\Phi_i(t) - \Phi_o(t)\right]dt .$$

The volume of the store at a point of time is equal:

$$W(t) = W_0 + \int_0^t \left[\Phi_i(t) - \Phi_o(t)\right]dt$$

where W_0 is the starting store of the unit.

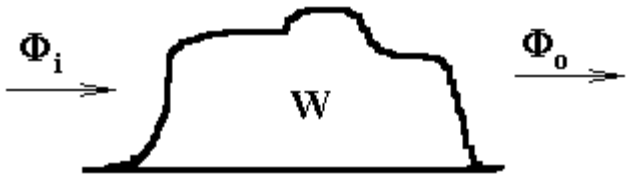


Figure no. 4. *Simple storing*

As we can see the simple store process has integral character.

2. Filling and draining of tanks

Tanks can be characterised as simple storage units, but in the everyday routine different parameters are used. The actual store is measured by the level of the liquid, which is proportional to the bottom pressure.

a. Open tank without outlet during filling

First the input flow is zero and the examined intensive parameter is the pressure on the bottom, which is proportional to the level of the liquid (h_0). The output mass flow is always zero. The cross section of the tank is constant, A.

$$\Phi_{vo}(t) = 0.$$

The constant mass flow, $\Phi_{vi1}(t) = \text{const.}$, starting at the zero point of the time will increase the level of the liquid as follows:

$$A\, dH = \Phi_{vi1}\, dt.$$

The canonical form of the state function is:

$$\frac{dh}{dt} = \frac{1}{A}\Phi_{bi1}.$$

We can get the liquid level vs. time curve by integration:

$$h(t) = \frac{1}{A}\int_0^t \Phi_{vi1} dt = \frac{1}{A}\Phi_{vi1} t + h_0.$$

The level of the liquid is proportional to the integral of the input mass flow. This elementary unit has integrating characteristic.
When the output flow is proportional to v, the parameter under examination:

$$\Phi_k = kv.$$

The canonical form of the dynamic state function is:

$$\frac{dv}{dt} = \frac{1}{C}\Phi_i(t) - \frac{k}{C}v.$$

The elementary units of dynamic state function like this are capacity delaying units.
How do the v parameter change when the input flow is changing by a given time function?
The starting value of the input flow and the examined parameter are zero. The $\Phi_{i1} = \text{const}.$ input flow starts at the $t = 0$ point. The name of this function is the step function, it is one of the typical probe functions. The stationer state of the elementary unit has changed and the intensive parameter, v begins changing.

$$\Phi_i(t) = \Phi_{i1} = \text{const.}$$

$$\frac{dv}{dt} = \frac{1}{C}\Phi_{i1} - \frac{k}{C}v.$$

After separation of variables:

$$\frac{dv}{\frac{1}{C}\Phi_{i1} - \frac{k}{C}v} = dt.$$

In integrated form:

$$-\frac{C}{k}\ln\left(\frac{1}{C}\Phi_{i1} - \frac{k}{C}v\right) = t + c_1.$$

In an other form:

$$\frac{1}{C}\Phi_{i1} - \frac{k}{C}v = c_2 e^{-\frac{k}{C}t}.$$

The integrating constant when $t = 0$, $v = 0$ is:

$$c_2 = \frac{1}{C}\Phi_{i1}.$$

Replacing this to the earlier equation:

$$\frac{1}{C}\Phi_{i1} - \frac{k}{C}v = \frac{1}{C}\Phi_{i1} e^{-\frac{k}{C}t}.$$

The time function of the v parameter is:

$$v(t) = \frac{\Phi_{i1}}{k}\left(1 - e^{-\frac{k}{C}t}\right).$$

What are the main properties of this function?

If $t \to \infty$, then $v \to \frac{\Phi_{i1}}{k} = \text{const.}$

After the disturbance the examined parameter is approaching to a new constant value, which is proportional to the new value of the disturbance. This new value on an exponential curve will be reached only in infinite time. The $v(t)$ is the step response function of the elementary unit. The delay of the process is characterised by the timing constant T. The numerical value of the timing constant means the time while the examined parameter would rich the new constant value with the changing speed at the beginning.

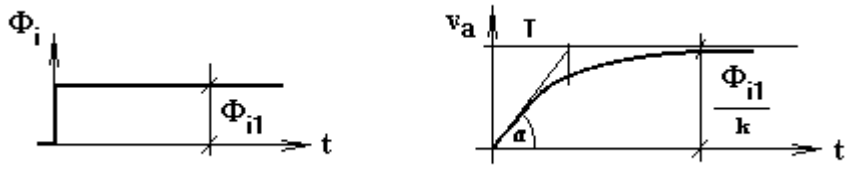


Figure no. 5. *The step response function of the tank*

$$\mathrm{tg}\alpha = \frac{\Phi_{i1}}{kT} = \left(\frac{dv}{dt}\right)_{t=0}.$$

At the same time:

$$\left(\frac{dv}{dt}\right)_{t=0} = \left(\frac{\Phi_{i1}}{k}\frac{k}{C}e^{-\frac{k}{C}t}\right)_{t=0} = \frac{\Phi_{i1}}{C}.$$

And:

$$T = \frac{C}{k}.$$

The time constant does not depend on the amount of input and output flows, and it is the only property of the elementary unit.
The elementary units like this are named to proportional unit with single capacitor, because in the constant state the output parameter is proportional to the input flow, but the new value after the disturbance is reached only after delay.

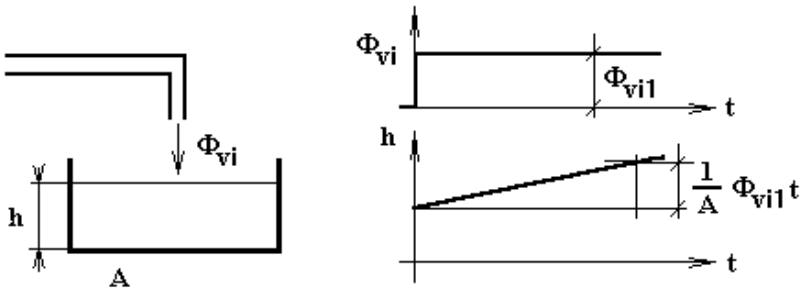


Figure no. 6. *Tank under filling without outlet*

b. Open tank under filling and draining

Let see the tank of cross section A, being filled by $\Phi_{vi}(t)$ mass flow and in the same time drained by $\Phi_{vo}(t)$. If $\Phi_{vi}(t)$ and $\Phi_{vo}(t)$ are independent, then the tank can be seen as a simple storing unit. The tank with an outlet at the bottom is a different case.

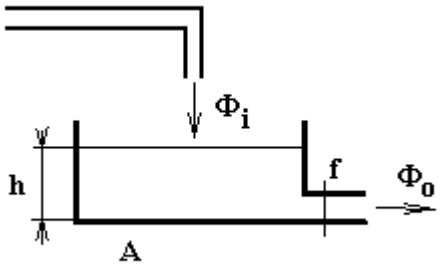


Figure no. 7. *Tank under filling and draining*

Find first the static state function of this tank! The liquid level / h / which is proportional to the bottom pressure is the intensive parameter. In stationer case the input and the output mass flows are equal to each other:

$\Phi_i = \Phi_o = \text{const.} = \Phi$.

and the level of the liquid will be h. The output mass flow depends on the level:

$\Phi = \mu f \sqrt{2gh}$,

where f is the cross section of the output pipe, μ is the output coefficient.
The solution of the equation is:

$$h = \left(\frac{\Phi}{mf\sqrt{g}} \right)^2 .$$

This is the static state function of a tank under filling and draining, it gives the relation of the liquid level and the input mass flow in permanent condition.
The dynamic state function of the tank can be derived on the next way: if $\Phi_i \neq \Phi_o$, then the balance will be ended and the difference of the input and the output mass flow will increase the level in the tank:

$(\Phi_i - \Phi_o)dt = A\ dh$,

where $\Phi_o = \mu f \sqrt{2gh}$,

and A is the constant cross section of the tank. Introduced the k designation:

$k = \mu f \sqrt{2g}$, and

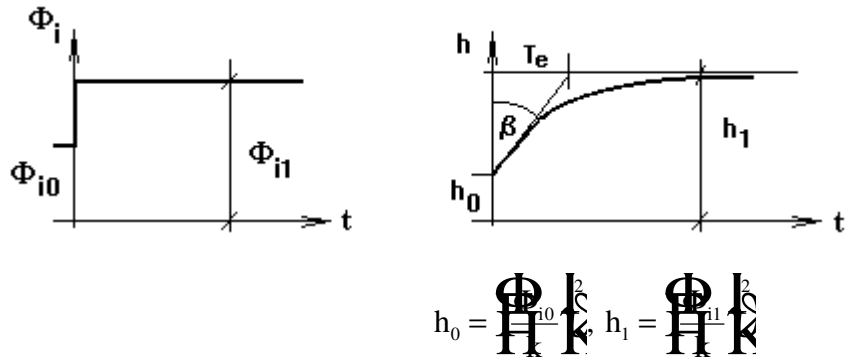
$\Phi_o = k\sqrt{h}$.

From these equations we can get the canonical form:

$$\frac{dh}{dt} = \frac{1}{A}\left(\Phi_i - k\sqrt{h}\right).$$

This is the dynamic state function of the tank under filling and draining. It can be seen, that this is a delay unit with capacity and the capacity coefficient is the cross section of the tank. This function is not a linear one, because the output flow is proportional not to the liquid level but to the square root of it. So the delay constant couldn't be established by the different equation, and the question is, could we define a time constant at all?
Let see the time function of the liquid level in the tank, started from the h_0 stabilised state. The input flow suddenly changes from Φ_{i0} to Φ_{i1}. In this case the differential equation will be the next:

$$\frac{dh}{dt} = \frac{1}{A}\left(\Phi_{i1} - k\sqrt{h}\right).$$



$$h_0 = \left(\frac{\Phi_{i0}}{k}\right)^2, \quad h_1 = \left(\frac{\Phi_{i1}}{k}\right)^2$$

Figure no. 8. *The step response function*

We can get the inverse of the $h = h(t)$ function easier:

$$\int_0^t dt = \int_{h_0}^{h} \frac{dh}{\frac{1}{A}\left(\Phi_{i1} - k\sqrt{h}\right)}.$$

The reduced equation is:

$$t = \frac{A}{k}\int_{h}^{h_0} \frac{dh}{\frac{\Phi_{i1}}{k} - \sqrt{h}}.$$

With the help of a new variable the integration can be realised:

$$u = \frac{\Phi_{i1}}{k} - \sqrt{h}.$$

$$t = \frac{2A}{k}\left(\sqrt{h_0} - \sqrt{h} + \frac{\Phi_{i1}}{k} \ln \frac{\frac{\Phi_{i1}}{k} - \sqrt{h_0}}{\frac{\Phi_{i1}}{k} - \sqrt{h}} \right).$$

The $h = h(t)$ function depends also on the starting state (h_0) of the tank, not only on the new input flow (Φ_{i1}) and the tank parameters (k, A). This comes from the non linearity of the elementary unit.
The step response function can be drawn on the basis of the last equation. The level of the liquid will increase up from the h_0 after raising of the input mass flow and the output flow will also increase with it. In the new stabilised state the $\Phi_{i1} = \Phi_{o1}$ relation comes true again. As we saw earlier,

$\Phi_{o1} = k\sqrt{h_1} = \Phi_{i1}$, so

$$\frac{\Phi_{i1}}{k} = \sqrt{h_1}.$$

The physical meaning of $\frac{\Phi_{i1}}{k}$ is the square root of the liquid level in the new stabilised state, so

$$t = \frac{2A}{k}\left(\sqrt{h_0} - \sqrt{h} + \sqrt{h_1} \ln \frac{\sqrt{h_1} - \sqrt{h_0}}{\sqrt{h_1} - \sqrt{h}} \right).$$

The step response function can be seen on right hand side of the figure 8.
The step response is not a simple exponential curve, the shape of it depends on the begin and the end stabilized state.
The time constant is the time, while the examined parameter would rich its new stable level with the maximum changing speed at the begin of the change. The derivative of the inverse function is:

$$\frac{dt}{dh} = \frac{A}{k} \frac{1}{\sqrt{h_1} - \sqrt{h}}$$

The starting slope of the tangent, if $t = 0$, $h = h_0$:

$$\left[\frac{dt}{dh} \right]_{t=0} = \frac{A}{k} \frac{1}{\sqrt{h_1} - \sqrt{h_0}} = \mathrm{tg}\beta .$$

On the other way

$$\mathrm{tg}\beta = \frac{T_e}{h_1 - h_0}, \text{ and}$$

$$T_e = \frac{A}{k}\frac{h_1 - h_0}{\sqrt{h_1} - \sqrt{h_0}} = \frac{A}{k}\left(\sqrt{h_1} + \sqrt{h_0}\right).$$

So the 'time constant' is not constant, it depends on the starting and the end level (h_0, h_1). For making difference T_e is named equivalent time constant. If we want to use the model of the tank with a real time constant instead of this non linear model, then we can calculate the speed of liquid changing at the start point only.
In a working point (h_0, Φ_{i0}) we can linearise the model. If the change is small, and

$$\frac{h_1}{h_0} \approx 1 ,$$

then the equivalent time constant in the working point is:

$$T_e = \frac{2A}{k}\sqrt{h_0} .$$

Reference:

Edited by Dr. Imre Szabó: Gépészeti rendszertechnika, Mûszaki Könyvkiadó, Budapest, 1986